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HONORS CALC II F 08, MIDTERM II

(1) Suppose that $\sum_{n=1}^{\infty} a_n$ is a convergent series and $0 < a_n < 1$ for all n. Show that $\sum_{n=1}^{\infty} \ln(1+(-1)a_n)$ is absolutely convergent.

(2) For each of the following series, determine whether it converges absolutely, converges conditionally or diverges.

(a) $\displaystyle\sum_{n=10}^{\infty} (-1)^n \frac{\ln(\ln n)}{\ln n}$ (b) $\displaystyle\sum_{n=1}^{\infty} \frac{(2n)!}{3^n (n!)^2}$ (c) $\displaystyle\sum_{n=1}^{\infty} (\sqrt{n + \frac{1}{\sqrt{n}}} - \sqrt{n})$.

(3) Find the Taylor expansion of the following functions.

(a) $f(x) = x \ln(1 + 2x)$ about $a = 0$ (b) $f(x) = \dfrac{1}{\sqrt{x}}$ about $a = 4$.

(4) Consider the polynomial $P(x) = 1 + x^2 + 2x^3 + x^4$. Write it as a polynomial in $(x-1)$. That is, find constants $c_0, \cdots, c_4$, such that $P(x) = c_0 + c_1(x-1) + c_2(x-1)^2 + c_3(x-1)^3 + c_4(x-1)^4$.

Solutions

(1) We know that $\lim_{x\to 0}\frac{\ln(1+x)}{x}=1$. Therefore for all x sufficiently small, $|\ln(1+x)|<2|x|$. Since $\lim_{n\to\infty}a_n=0$, it follows that for all n large enough $|\ln(1+(-1)^n a_n)|<2a_n$, from which the claim immediately follows.

(2)

(a) Let $b_n=\frac{\ln\ln n}{\ln n}$. Then for n large $|b_n|>\frac{1}{\ln n}$. This implies that the series does not converge absolutely.
Consider the function $f(x)=\frac{\ln x}{x}$. Note that $\lim_{x\to\infty}f(x)=0$. Also $f'(x)=\frac{1}{x^2}-\frac{\ln x}{x^2}=\frac{1-\ln x}{x^2}$. In particular, f is decreasing in x for all $x>e$. Letting $x=\ln n$ and using the fact that $\ln(n+1)>\ln n$ for all n, and that $\lim_{n\to\infty}\ln n=\infty$, we see that b_n decreases to 0 as $n\to\infty$. This guarantees convergence by the alternating series test. Hence the series converges conditionally. (b) Use ratio test: $a_{n+1}/a_n=\frac{(2n)(2n+2)}{3(n+1)^2}\to\frac{4}{3}$. Therefore the series diverges.

(c) Note that

$$a_n\frac{\sqrt{n+\frac{1}{\sqrt{n}}}+\sqrt{n}}{\sqrt{n+\frac{1}{\sqrt{n}}}+\sqrt{n}}=\frac{n^{-1/2}}{\sqrt{n+\frac{1}{\sqrt{n}}}+\sqrt{n}}=\frac{1}{2n\left(\sqrt{1+n^{-3/2}}+1\right)}.$$

Thus the series diverges by comparison with the harmonic series.

(3)

(a)

$$\ln(1+t)=\sum_{n=1}^{\infty}(-1)^{n+1}\frac{t^n}{n}.$$

Set $t=2x$ to obtain

$$f(x)=x\sum_{n=1}^{\infty}(-1)^{n+1}\frac{(2x)^n}{n}=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}2^n}{n}x^{n+1}=\sum_{k=2}^{\infty}\frac{(-1)^k 2^{k-1}}{k-1}x^k.$$

(b) $f(x)=x^{-1/2}$. Looking at the first two derivatives we see that $f'(x)=(-\frac{1}{2})x^{-3/2}$, $f''(x)=(-\frac{1}{2})(-\frac{3}{2})x^{-5/2},\cdots$. Thus by simple induction $f^{(n)}(x)=\frac{(-1)^n\cdot 1\cdot 3\cdots(2n-1)}{2^n}x^{-(2n+1)/2}$. Therefore $f^{(n)}(4)=\frac{(-1)^n\cdot 1\cdot 3\cdots(2n-1)}{2^{3n+1}}$, and the Taylor series for f about 4 is

$$f(x)=\sum_{n=0}^{\infty}\frac{(-1)^n\cdot 1\cdot 3\cdots(2n-1)}{2^{3n+1}n!}(x-4)^n.$$

If you wish to present it in a somehow shorter form, note (as we did in class) that $1\cdot 3\cdot(2n-1)=\frac{(2n)!}{2^n n!}=\binom{2n}{n}2^{-n}$. Therefore after doing the algebra we get

$$f(x)=\frac{1}{2}\sum_{n=0}^{\infty}\frac{(-1)^n}{16^n}\binom{2n}{n}(x-4)^n.$$