Analysis I (Math 3150) Fall 2009
Schedule and Homework

Tentative Schedule

Weeks are numbered chronologically with no skipping, starting from the first week of the semester.

Week	Sections	Comments	Material Actually Taught
1	1.1		1.1
2	1.3,1.4	Quiz 1	1.3, began 1.4
3	2.1,2.2		1.4, began 2.1
4	2.3,2.4	Quiz 2	2.1,2.2 (except: Th. 2.9,2.17)
5	2.5,3.1,3.2	Midterm I	2.3 - Monotone seq, 2.4
6	3.3,3.4	Quiz 3	Finished 2.3,2.5
7	4.1,4.2		3.1,3.2 began 3.3
8	4.3,4.4	Quiz 4	finished 3.3, began 3.4
9	5.1,5.2,5.3		4.1,4.2
10	5.4,5.5*,5.6*	Quiz 5	4.3,4.4
11	6.1,6.2	Midterm II	Began 5.1
12	6.3,6.4,6.6	Quiz 6	Finished 5.1
13		Thanksgiving	
14	7.1,7.2,7.3	Quiz 7	5.3,6.1,6.2
15	7.4,(8.1-8.4)*		6.3,6.4

Starred sections will be covered time permitting.

Homework Assignments

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HW 1 (posted Week 1, 09/03)

In all problems below, F is a field and $a \in F$.

1. Prove that $(-1)a = (-a)$.
2. Let $\mathbb{N}_F = \{1, 1+1, 1+1+1, \dots\} \cap \{0\}^c$, $\mathbb{Z}_F = \{a \in F : a \in \mathbb{N} \text{ or } -a \in \mathbb{N}\} \cup \{0\}$. Let $\mathbb{Q}_F = \{pq^{-1} : p \in \mathbb{Z}_F, q \in \mathbb{N}_F\}$. Prove that $\mathbb{Q}_F$ is a field, and it is the minimal field in F: $\mathbb{Q}_F \subseteq G$ for any other field in G contained in F (the operations for $G, \mathbb{Q}_F$ are induced by the operations in F).

   In all problems below, assume that F is totally ordered.

3. Prove that if $0 < a < b$, then $\frac{1}{b} < \frac{1}{a}$.
4. Prove that $|a| = |-a|$ and $a \leq |a|$.
5. Let $M \in F$ be positive and let $a \in F$. Prove that $|a| < M$ if and only if $-M < a < M$.
6. Complete the proof of Theorem 1.9.
7. Show that $1 + \cdots + 1 \neq 0$ and conclude that any totally ordered field is infinite.

Solution

1. $0 = 0a = (1 + (-1))a = a + (-1)a$. So $(-a) = (-1)a$.
2. Any field contained in F must clearly contain $\mathbb{Q}_F$. This proved the second claim. Now it is a straightforward computation that $\mathbb{Q}_F$ is closed under addition, multiplication, additive and multiplicative inverses. Therefore it is a field.
3. $0 < a < b$. So $0 < (b - a)$. Now for $c > 0$ we have $c^{-1} > 0$. Otherwise $c^{-1} \leq 0$ and then $1 = c^{-1}c \leq 0$ (why ?), a contradiction. In particular, $a^{-1}b^{-1} > 0$. Thus $b^{-1} = aa^{-1}b^{-1} < ba^{-1}b^{-1} = a^{-1}$.
   (Note this statement is an if and only if one)
4. Without loss of generality $a \geq 0$, and so $-a \leq 0$. Now $|a| = a$ and by definition $|-a| = -(-a) = a$. Next, if $0 \leq a$, then $a = |a|$ so $a \leq |a|$. Otherwise $a \leq 0$ and so $a \leq 0 \leq -a = |a|$ and the claim follows.
5. Claim is obvious for nonnegative a. Now if $a \leq 0$, then $0 \leq |a| = -a$, and so $-a = |a| < M$ if and only if $-M < a$ (add $a - M$ to both sides for forward implication and subtract for reverse).
6. In the book.
7. $0 < 1 < 1 + 1 < 1 + 1 + 1, \dots$ (using the fact that the relation $\leq$ is preserved when adding the same number to both sides). Therefore each new member of this sequence of sums of 1's is different from all the previous ones. This shows both statements.

HW 2 (posted Week 2, 09/10)

Section 1.3:2,3,4,8,9,10

Solution

2 Let $M = \sup E$. Then there exist $n \in \mathbb{Z}$ such that $n \leq M < n+1$ (discussed in class, see below) Now if $e \in E$, then $e < n+1$, so $e \leq n$. Hence $n = \sup E$, so $M = n$.
To prove the part it is enough to show that for any set $A \subset \mathbb{R}$, $m = \inf A$ if and only if $-m = \sup(-A)$, where $-A = \{-x : x \in A\}$. I leave this to you, and the result then follows.

3 We know that $\sqrt{2}$ is irrational. Indeed, if $\sqrt{2} = p/q$ for relatively prime p, q, then $2q^2 = p^2$. Hence p^2 is an even square, so it has to be divisible by 4. But then $q^2 = \frac{p^2}{2}$ is also an even square, so it is divisible by 4, violating the assumption that p and q are relatively prime. Also note that $\sqrt{2} < 2$ (why ?) Next, note that x is irrational then $x + r$ and xr are irrational for all $0 \neq r \in \mathbb{Q}$. Argue by contradiction. If xr is rational. Then $x = (xr) * r^{-1}$ is a product of rational and so is also rational. Similarly, if $x+r$ is rational, then so is $x = x+r-r$. Next let $q \in (a, b)$ be rational. We know there exists one, and let $\epsilon = \frac{\sqrt{2}}{2n}$ for n satisfying $(b-q) < n$. Then, $q + \epsilon$ is irrational $> a$ and $< q + \epsilon < q + (b-q) < b$.

4 Take a rational r in $[a, a+1/n]$, and the claim follows.

8 The set $E_n = \{x_n, x_{n+1}, \dots\}$ is bounded. Therefore $s_n := \sup E_n$ exists and $s_n \leq M$. Now any upper bound of the "larger" set E_n is an upper bound of the "smaller" set E_{n+1}. In particular, s_n is an upper bound for E_{n+1}, therefore $s_n \geq s_{n+1}$. For other statement either apply the argument from [2], or repeat this argument with the necessary modifications.
Return to this after we discuss $\limsup$ and $\liminf$.

9 There exists some n such that $0 < 2 < (b-a)n < (b-a)10^n$ because $10^n > n$. Now This implies that $1 < 10^n a < 10^n a + 2 < 10^n b$. Therefore there exists $m \in \mathbb{N}$ in $[10^n a + 1, 10^n b)$. Call it m. Then $10^n a < m < 10^n b$. Divide both by 10^n, and claim is proved.

10 $A \subset A \cup B$, therefore any upper bound of $A \cup B$ is an upper bound for A, in a particular, if $A \cup B$ has a supremum, then it's bounded, and therefore also A, and the result follows by completeness. Now $M = \max(\sup A, \sup B)$ is an upper bound for $A \cup B$, while if $x \leq M$, then there exists $a \in A$ such that $x \leq a$ or $b \in B$ such that $x \leq b$. Therefore x is not an upper bound. It follows that M is the least upper bound for $A \cup B$.

HW 3 (posted Week 3, 09/17)

1. Let A, B be nonempty sets. Prove that there exists an onto function $f : A \to B$, if and only if there exists a 1-1 function $g : B \to A$.
2. Prove Theorem 1.37-(i)(ii). Note: our notion of countable is what the book calls "at most countable" (our notion of countable includes finite sets).
3. Suppose that $A \subset B \subset \mathbb{R}$, and that A is countable, and B is not. Then $B \backslash A = B \cap A^c$ is uncountable.
4. Prove that any interval $I \subset \mathbb{R}$ with endpoints $a < b$ is uncountable.
   (Hint: Let $h : \mathbb{R} \to \mathbb{R}$ be the mapping $h(x) = 4(x-a)/(b-a) - 2$. Observe that it is one to one and maps I onto an interval containing $[0,1]$.
5. Let $C \subset [0,1]$ be the set containing all numbers whose decimal expansion contains only the digits 2 and 7. Is this set countable ? Does it contain any interval ?
6. Sec. 1.4: Problems 5,6,9
7. (*) Let $x \in (0,1)$ be irrational. Show that if $0 \leq a < b \leq 1$, then there exists $m \in \mathbb{N}$ such that $a < (mx \mod 1) < b$ (recall that $x \mod 1$ is equal to x minus its integer part).
   (Hint: Divide $(0,1)$ into n disjoint intervals of length $\frac{1}{n}$. Look at the first $n+1$ iterations $x, 2x \mod 1, 3x \mod 1, \ldots, (n+1)x \mod 1$, and observe that at least two are in the same interval. This is called the Pigeonhole Principle: if you have more pigeons than pigeonholes, you will find two pigeons in the same hole).

Solution

1. $\Rightarrow$ Each $b \in B$ is equal to $f(a)$ for some (not necessarily unique !) $a \in A$. Choose such an arbitrary a, and let $g(b) = a$. Now $g : B \to A$. Note that if $g(b) = a = g(b')$, then $b = f(a) = b'$, therefore $b = b'$. Hence g is 1-1.
   $\Leftarrow$ For each $a \in A$ in the image of g, there exists a unique $b \in B$ such that $g(b) = a$, for such a, let $f(a) = b$. This already defined an onto function f : Image of $g \to B$. Finally, for a not in the image of g, let $f(a) = b_0$ where b_0 is some arbitrary fixed element of B.
2. Check book.
3. Union of two countable sets is countable. If both A and $B \backslash A$ are, then their union, B is also, a contradiction. (**typo correction, 10/01 03:10PM**)
4. Well h is clearly one to one, and it maps the interval (a,b) (contained in I) onto $(-2,2)$, which contains $[0,1]$. Hence if $f : \mathbb{N} \to I$ is onto, then $h \circ f$ maps $\mathbb{N}$ onto some interval containing $[0,1]$, contradiction to the uncountability of $[0,1]$.
5. No to both questions. We can't list them, by the very same argument we used to show that the set of infinite sequences of $\{0,1\}$ is uncountable. Next, this set does not contain any interval because if it does, then there's some x and n such that $x, x + 10^{-n}$ are both in this set. However, $x + 10^{-n}$ can't be in that set because it must contain either the digit 3 or 8.
6.
   5 (a) $[-1,2]$. (b) $[0,1]$ (c) $[0,1]$ (d) $\{0\}$ (**correction, 08/29 08:53AM**)

   6 $(a) \Rightarrow (b)$ The right-hand side is the image of all elements in A which are not also images of elements in B. Since f is 1-1, this is exactly the image of all elements in A which are not in elements in B, proving the equality.
   $(b) \Rightarrow (c)$ Suppose not. Then $f^{-1}(f(E))$ contains E and an element not in E, say b. Let $B = \{b\}$. Then $f(E) = f(0E \backslash B) = f(E) \backslash f(B) = f(E) \backslash \{b\}$, a contradiction.
   $(c) \Rightarrow (d)$ Clearly $f(A \cap B)$ is a subset of $f(A)$ and a subset of $f(B)$, so it's contained in $f(A) \cap f(B)$. Now if $w \in f(A) \cap f(B)$, then let $a \in A$, $b \in B$ such that $f(a) = f(b) = w$. By (c), $a = f^{-1}(f(\{a\})) = f^{-1}(f(\{b\})) = b$. In particular, $a, b \in A \cap B$, proving $w \in f(A \cap B)$.
   $(d) \Rightarrow (a)$ Let $A = \{a\}, B = \{b\}$. Then by (d), $A \cap B$ is empty if and only if $f(a) \neq f(b)$. But $A \cap B$ is empty if and only if $a \neq b$. Therefore, $f(a) \neq f(b)$ if and only if $a \neq b$.

9 (a) Assume both 1-1. Then if $a \neq a'$, then $f(a) \neq f(a')$ and so $g(f(a)) \neq g(f(a'))$ hence $g \circ f$ is 1-1. Assume both onto, then for each $c \in C$ there exists $b \in B$ such that $g(b) = c$ and for each $b \in B$ there exists $a \in A$ such that $f(a) = b$. Thus, $g(f(a)) = c$, proving $g \circ f$ is onto.

(b) For each $b \in f(A)$, there exists a unique $a \in A$ such that $f(a) = b$. Define $f^{-1}(b) = a$. Then f^{-1} is a one to one function from $f(A)$ onto A.

Note: this exercise is not well-worded. In general, f^{-1} NOT a function on B because it maps subsets (not elements) of B to subsets of A. Therefore, what we do here is abuse of notation. In fact we do it with all inverse functions. (c) If f is 1-1, then by (a), $g \circ f$ is 1-1. Conversely, if $g \circ f$ is 1-1 and $a \neq a'$, then $g(f(a')) \neq g(f(a))$, therefore $f(a') \neq f(a)$.

7. You may want to recall some facts above irrational numbers as listed on the solution to Problem 1.3,3 from HW 2. Let $a_m = mx \mod 1$. First $a_m = mx - l$ for some nonnegative integer l, therefore a_m is irrational. By definition, $a_m \in [0, 1]$. Choose n large enough to satisfy $\frac{1}{n} < \frac{b-a}{2}$. According to the hint, there exists $k \in \{0, \ldots, n-1\}$ and $1 \leq k_1 < k_2 \leq n+1$ such that $a_{k_1}, a_{k_2} \in [k/n, (k+1)/n]$. Without loss of generality, assume that $a_{k_1} < a_{k_2}$. Let $x_1 = a_{k_2} - a_{k_1} = (k_2 - k_1)x \mod 1 \neq 0$ (because $(k_2 - k_1)x$ is irrational and therefore not an integer). Then $x_1 \in [0, 1/n] < b - a$. In particular, there exists some nonnegative integer m such that $mx_1 \in (a, b)$. Now $x_1 = m((k_2 - k_1)x \mod 1)$. But for any real z and integer l, $l(z \mod 1) \mod 1 = lz \mod 1$. In particular, $mx_1 = a_{m(k_2 - k_1)}$, completing the proof.

Quiz 2 (Week 4, 09/24)

1. Define "a countable set". Let $A \subset \mathbb{R}$ be nonempty. Assume that for each $M \in \mathbb{R}$ the set $B_M = \{x \in A : x \leq A\}$ is finite. Prove that
   (a) A is bounded below.
   (b) A is countable.

Solution

1. Check your notebook.

2. (a) Fix any $M \in \mathbb{R}$. If B_M is empty, then let $m = M$. Otherwise, B_M has a finite number of elements, in particular it is bounded from below by some $m \in \mathbb{R}$. Let $x \in A$. Then either $x \in A \backslash B_M$ and then, $x > M \geq m$ or $x \in B_M$, and then $M \geq x \geq m$. In both cases, m is lower bound for A.
   (b) Note that $A = \cup_{M \in \mathbb{N}} A_M$, a countable union of finite (therefore countable) sets. Hence A is countable.

HW 4 (posted Week 4, 09/26)

Sec. 2.1:1cd,2bc,3,4,5b,6
Sec. 2.2:4,5,7 (all those do not require the theorems not discussed in class yet).

Solutions

2.1:1 (c) note that $0 < (5+n)/n^2 = \frac{5}{n^2} + \frac{1}{n} < \frac{2}{n}$. Fix $\epsilon > 0$. There exists $N(\epsilon)$ such that $2 < \epsilon n$ for all $n \geq N(\epsilon)$, and the claim follows.
(d) $a_n = \pi - \frac{3}{\sqrt{n}}$ so $|\pi - a_n| = \frac{3}{\sqrt{n}}$. Find $N(\epsilon)$ such that $9 < \epsilon^2 n$ for all $n \geq N(\epsilon)$ and so $|\pi - a_n|^2 < \epsilon^2$. Clearly, $|\pi - a_n| < \epsilon$.

2.1:2 (b) $|3x_n + 1 - 4| = |3x_n - 3| = 3|x_n - 1|$. By definition, $x_n \to 1$ if and only if $|x_n - 1| \to 0$
(c) $|(2 + x_n^2)/x_n - 3| = |(x_n^2 - 3x_n + 2)/x_n| = |\frac{(x_n-1)^2+(1-x_n)}{x_n}| \leq \frac{|x_n-1|^2}{|x_n|} + \frac{|x_n-1|}{|x_n|} = (*)$. For $n \geq N(\min\{\epsilon/3, \frac{1}{2}\})$, $|x_n - 1| < \frac{1}{2}$ and $|x_n| = |1 + x_n - 1| \geq 1 - |x_n - 1| > \frac{1}{2}$. Therefore $(*) < |x_n - 1| + 2|x_n - 1| = 3|x_n - 1| < 3\epsilon/3 = \epsilon$.

2.1:3 Let $n_k = 2k$. Then (x_{n_k}) is the constant sequence $1, 1, 1, \ldots$ which converges. Next, the sequence itself does not converge and this is clearly a non-convergent subsequence. If you wish a proper subsequence, take $n_k = 3k$. Then (x_{n_k}) is the sequence $-1, 1, -1, \ldots$ (the original sequence), which we have shown to be non-convergent.
(b) When n is odd, $(-1)^{3n} = (-1)^n = -1$. Therefore letting $n_k = (2k-1)$ for $k = 1, \ldots,$ (all odd indices, the resulting subsequence (x_{n_k}) is the constant sequence $(0, 0, 0, \ldots)$ which obviously converges to 0.

2.1:4 (a) Let $\epsilon > 0$ be given then for $n \geq N(\epsilon)$, $|b_n - 0| = |b_n| < \epsilon$. In particular, $|x_n - a| < |b_n| < \epsilon$, completing the proof.
(b) Choose $N(\epsilon/C)$ in part (a) and so $|x_n - a| < C\epsilon/C = \epsilon$ for $n \geq N$.

2.1:5 (a) Recall $-|x_n| < x_n < |x_n|$. Therefore, if $|x_n| < C$, it follows that $x_n \in (-C, C)$. Thus, C is an upper bound and $-C$ is a lower bound. Conversely, if the sequence is bounded, below by m and above by M, then $x_n \leq M \leq |M| \leq |M| + |m|$. Similarly, $x_n \geq m \geq -|m| \geq -|m| - |M|$. so $|x_n| < |M| + |m|$ for all n. (b) Let C be as above. For $n \geq \frac{C}{\epsilon}$ we have $|x_n - 0| \leq \frac{C}{n^k} < \frac{C}{n} < C\epsilon/C = \epsilon$.

2.1:6 (a) Let $a = \lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n$. Let N be such that for $n \geq N$, $|x_n - a| < \epsilon/2$, and $|y_n - a| < \epsilon/2$. Then $|x_n - y_n| = |x_n - a + a - y_n| < |x_n - a| + |y_n - a| < \epsilon$.
(b) A convergent sequence is bounded. This is not, so it can't be convergent. Why ? Let $M > 0$. Then there exists n such that $M < n\dot{1}$. In order words, for M there exists some element in the sequence bigger than M. So no M is an upper bound.
(c) Take $x_n = y_n = n$. Then $x_n - y_n = 0$, a convergent sequence, but both (x_n) and (y_n) do not converge.

2.2:4 Let $z_n = |\sqrt{x_n} - \sqrt{x}|$. Then $z_n(\sqrt{x_n} + \sqrt{x}) = |x_n - x|$. Split into two cases. First $x > 0$. Then $z_n\sqrt{x} < z_n(\sqrt{x_n} + \sqrt{x}) < |x_n - x|$. So $z_n < \frac{1}{\sqrt{x}}|x_n - x|$ and the claim follows (take $n \geq N(\sqrt{x}\epsilon)$). If $x = 0$, $z_n = \sqrt{x_n}$, and letting $N(\epsilon^2)$ we have $|x_n - 0| < \epsilon^2$ and in particular, $\sqrt{x_n} < \epsilon$.

2.2:5 Let $r_n \in (x, x + \frac{1}{n})$ be rational (between two numbers there's a rational). Now $|r_n - x| < \frac{1}{n}$. This settles the proof (see 2.1:4).

2.2:7 If $x_n \to x$, then $(x_n + 2) \to x + 2$ and by 4, $\sqrt{x_n + 2} \to x + 2$. Now (x_{n+1}) is a subsequence of (x_n) $(n_k = k + 1)$, therefore converges to x. We then must have $x = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} \sqrt{x_n + 2} = \sqrt{x + 2}$. So $x^2 = x + 2$ or$x^2 - 4 = (x - 2)$. But $x^2 - 4 = (x - 2)(x + 2)$. Hence $(x - 2)(x + 2) = (x - 2)$. Thus, $x = 2$ or $x + 2 = 1$ and then $x = -3$. But $|x_n - (-3)| > 3$ for all n because $x_n \geq 0$. Hence $x = 2$.
(b) Same argument shows $x = 1 - \sqrt{1 - x}$. Therefore $1 - x = \sqrt{1 - x}$ or $(\sqrt{1 - x})^2 = \sqrt{1 - x}$. So either $x = 1$ or $x = 0$.

Midterm I (Week 5, 10/01)

1. Let E be a nonempty subset of $\mathbb{R}$. Define:
   (a) E is bounded above.
   (b) (Assuming E is bounded above) M is the supremum of E.

2. Let (a_n) be a sequence of real numbers, all different from 0, satisfying $\lim_{n\to\infty} a_n = \infty$.
   (a) Prove that $\lim_{n\to\infty} \frac{1}{a_n} = 0$.
   (b) Is the converse true (that is, does the condition $\lim_{n\to\infty} \frac{1}{a_n} = 0$ imply $\lim_{n\to\infty} a_n = \infty$) ?

3. (a) Prove that there exists a sequence (r_n) consisting only of rationals and containing all rationals (that is $r_n \in \mathbb{Q}$ for all n and for each $q \in \mathbb{Q}$ there exists n such that $r_n = q$).
   (b) Let (r_n) be the sequence from part (a). Prove that for each $x \in \mathbb{R}$ there exists a subsequence (r_{n_k}) converging to x (you are allowed to use the fact that the rationals are dense in the real numbers).

4. Determine whether the each of the following is true or false. If true, prove. If false, find a counterexample.
   (a) Let E, F be subsets of $\mathbb{R}$ such that $\sup E < \inf F$. Prove that $E \cap F = \emptyset$.
   (b) If $E \subset \mathbb{R}$ is such that $\sup E$ exists, then $\sup E \in E$.
   (c) Every uncountable subset of $[0,1]$ contains an interval (a,b) for some $0 < a < b < 1$.
   (d) If (a_n) is a convergent sequence of nonnegative numbers then $\lim_{n\to\infty} a_n \geq 0$.

Solutions

1. Read notes.

2. (a) By definition for every $M \in \mathbb{R}$ there exists $N(M)$ such that if $n \geq N(M)$ $a_n > M$. In particular, fix $\epsilon > 0$ and let $M = \frac{1}{\epsilon}$. Then, for $n \geq N(\frac{1}{\epsilon}$ we have $a_n > \frac{1}{\epsilon}$. In particular, $a_n > 0$. Multiply both sides by the positive number ϵ / a_n to obtain $\epsilon > \frac{1}{a_n} = |\frac{1}{a_n} - 0|$, completing the proof.
   (b) No. Let $a_n = (-1)^n n$. Then $|\frac{1}{a_n} - 0| = \frac{1}{n}$ and so $\lim_{n\to\infty} \frac{1}{a_n} = 0$. However, given any $M \in \mathbb{R}$, there exists $N \in \mathbb{N}$ such that $-N < M$ and so for all $n \geq N$, $a_{2n+1} = -(2n+1) < -n < M$, contradicting definition of convergence to ∞.

3. (a) $\mathbb{Q}$ is countable. That is there exists $f : \mathbb{N} \to \mathbb{Q}$ which is onto. Let $r_n = f(n)$. This sequence attains only rational values and attains all rational numbers.
   (b) By density of rationals we mean that between any two real numbers $a < b$, there exists a rational. Let's start. Let $a = x, b = x+1$. By density, there exists a rational $r \in (x, x+1)$. Let n_1 be such that $r_{n_1} = r$. Continue by induction, assuming that r_{n_k} is a rational bigger than x. The next element in the subsequence will be a rational less that "halfway" from x to r_{n_k}, that is in the interval $(x, x + (r_{n_k} - x)/2)$. How ? Well, note that there are infinitely many rationals in this interval. Otherwise, there's only a finite number and then here exists a minimal one, but then by density, between x and it there has to be another rational, contradiction to minimality. Now since the set $\{r_1, r_2, \ldots, r_{n_k}\}$ is finite, one of the infinitely many rationals in that interval is given by r_n for some $n > n_k$. Choose the smallest such n as n_{k+1}. Clearly, $n_{k+1} > n_k$. This completes the construction of the subsequence. It remains to prove that (r_{n_k}) converges to x. By definition, $|r_{n_{k+1}} - x| > |r_{n_k} - x|/2$ (being less than halfway) so by induction, $|r_{n_k} - x| < |r_1 - x|/2^{k-1}$. Since the sequence on the right hand side converges to 0 as $k \to \infty$, it follows that $r_{n_k} \to x$, as $k \to \infty$, completing the proof.

4. (a) TRUE. Let $x \in E$ and let $y \in F$. Then $x \leq \sup E < \inf F \leq y$, therefore $x < y$. So no two elements $x \in E$, $y \in F$ satisfy $x = y$. In order word $E \cap F = \emptyset$.

(b) FALSE. Let $E = (-\infty, 1)$. Then $\sup E = 1$ because 1 is an upper bound to E (by definition), and any smaller number x is not an upper bound to E (because for example, there exists a rational in $(x, 1)$). In particular, $1 = \sup E$, but $1 \notin E$.

(c) FALSE. An explicit example is provided in HW3, problem 5. Here's another method. $[0, 1]$ is uncountable, and $\mathbb{Q}$ is countable (both proved in class). Now $[0, 1]\backslash\mathbb{Q}$ is uncountable (HW 3, Problem 3), and does not contain an interval (because any interval (a, b) contains a rational by density of rationals).

(d) True. Argue by contradiction: suppose that some sequence (a_n) of nonnegative numbers has a strictly negative limit $-L < 0$. Then (taking $\epsilon = L/2$) for $n \geq N(L/2)$, $|a_n - (-L)| < L/2$. In particular $a_n - (-L) < L/2$ so adding $-L$ to both sides we have $a_n < -L/2$, or $2a_n < -L$, a contradiction, because $2a_n \geq 0 > -L$.

HW5 (posted Week 5, 10/02)

1. Suppose that (a_n) is a sequence converging in the extended sense and there exists $N \in \mathbb{N}$ such that $a_n \geq 0$ for all $n \geq N$. Prove that $\lim_{n\to\infty} a_n \geq 0$ (Midterm I, Problem 4(d)). Conclude:

   (a) (Monotonicity of limits). If (x_n) and (y_n) are sequences converging in the extended sense and there exists some $N \in \mathbb{N}$ such that $x_n \leq y_n$ for all $n \geq N$, then $\lim_{n\to\infty} x_n \leq \lim_{n\to\infty} y_n$.

   (b) (Squeeze theorem). Suppose that (x_n) and (z_n) are sequences converging in the extended sense, satisfying $\lim_{n\to\infty} x_n = \lim_{n\to\infty} z_n$. Let (y_n) be a sequence such that there exists $N \in \mathbb{N}$ and $x_n \leq y_n \leq z_n$ for all $n \geq N$. Prove that (y_n) iconverges in the extended sense and that $\lim_{n\to\infty} y_n = \lim_{n\to\infty} x_n$.

   Note: since $\pm\infty$ are not numbers, one needs to extend the notion of total order $\leq$ to extended numbers. This is done by requiring $-\infty \leq a \leq \infty$ for all extended numbers. a. This preserves all properties of total order. As before, for extended real numbers we write $a < b$ if $a \leq b$ and $a \neq b$. Note that a statement of the form $a \leq \infty$ (for an extended number a) is a tautology, and so if (x_n) converges in the extended sense, then there is no need to "prove" $\lim_{n\to\infty} x_n \leq \infty$, because the left-hand side is an extended real number. However, a statement like $\lim_{n\to\infty} x_n < \infty$ needs a proof, because one still needs to show that the left-hand side is not $+\infty$.

2. Prove that if (a_n) is an unbounded non-decreasing sequence then $\lim_{n\to\infty} a_n = \infty$.

3. Consider the following sequences. In each case, determine whether the sequence converges, converges in the extended sense, or neither. In the former two, find the limit.

   (a) $a_1 = 1$, $a_{n+1} = \frac{1}{1+\frac{1}{a}\,n}$. (Hint: (a_n) is non-decreasing).

   (b) $a_1 = 1,\ a_{n+1} = a_n + \frac{1}{n+1}$ (Hint: $\frac{1}{k} + \cdots + \frac{1}{2k} > \frac{1}{2}$).

   (c) $a_1 = 1,\ a_{n+1} = a_n + (-1)^n(n+1)$.

4. Let $P(x) = p_0 + \ldots p_n x^n$ and $Q(x) = q_0 + \cdots + q_m x^m$ be polynomials of degrees n, m respectively (in particular $p_n q_m \neq 0$). Let $a_k = P(k)/Q(k)$. Prove

$$\lim_{k\to\infty} a_k = \frac{p_n}{q_m}\begin{cases} \infty & n > m \\ 1 & n = m \\ 0 & n < m \end{cases}$$

5. Suppose that $\lim_{n\to\infty} a_n = \infty$. Prove, using only definitions, that (a_n) is not Cauchy.

6. Let (a_n) be a sequence. Suppose that the sequence (b_n), defined by $b_n = a_{n+1} - a_n$ is a Cauchy sequence. Is (a_n) necessarily Cauchy ?

7. Let (a_n) be a sequence. Prove that a_n is Cauchy if and only if $\lim_{n\to\infty} \max\{|a_n - a_k| : k > n\} = 0$.

8. (*) Limits of limits... A subset $E \subset \mathbb{R}$ is called closed if it's empty, or that whenever (x_n) is a convergent sequence consisting only of elements in E, then the limit is also in E (so it's closed under the operation of taking limits).
   Let (a_n) be a sequence. Let S denote the collection of all limits of all convergent subsequences of (a_n). Prove that S is closed.
   (*) Conversely, assume that S is any closed set. Show that there exists a real sequence (a_n) such that S is the set of all limits of all convergent subsequences of (a_n).

9. (*) Roots, etc...
   So far, the only operations we did with real numbers were combinations of addition and multiplication of numbers and their inverses. We have all heard of roots and more complicated operations. In this exercise we define roots and rational powers of positive numbers.
   Assume that $0 < y$. The positive n-th order root of y is a solution $x \geq 0$ to the equation $x^n = y$. We will show that a solution to this equation exists and is unique, therefore the notion of n-th order root is well defined. We denote the n-th order root of y by $y^{1/n}$. Having completed this, we define the notion of a rational power of y and study some of its basic properties. Let's start.

Step 1: Existence and uniqueness for root when $y \in (0,1]$.

- i. Let $A = \{0 \leq u \in \mathbb{R} : u^m \leq y\}$. Prove that A is nonempty and is bounded above by 1.
- ii. Let $x = \sup A$. Prove that $x^n = y$. You may use the inequality $(x+\eta)^n \leq x^n + (n+1)!\eta$, valid for all $x, \eta \in [0,1]$ (you can prove this inequality using the binomial formula).
- iii. Show that if $x' \leq 0$ and $(x')^n = y$, then $x' = x$.

Step 2: Existence and uniqueness of positive n-th order root for all $y > 0$.

- i. Prove that $x > 0$ satisfies $x^n = y$ if and only if $(\frac{1}{x})^n = y$.
- ii. Conclude that for any $y \geq 0$ the exists a unique positive n-th order root.

Step 3: Power Properties.
Recall that for any two integers, n_1, n_2, $y^{n_1+n_2} = y^{n_1}y^{n_2}$ and $(y^{n_1})^{n_2} = y^{n_1 n_2}$ (it's important to remember the following conventions: $y^0 = 1$ and for $n \in \mathbb{N}$, y^{-n} is a shorthand notation for $(y^n)^{-1}$). We extend these properties to rational powers of y. First we have to define what rational powers are. For $y > 0$, denote the positive n-root of y by $y^{1/n}$.

- i. Prove that for $n_1, n_2 \in \mathbb{N}$. $(y^{1/n_1})^{1/n_2} = y^{1/(n_1 n_2)}$.
- ii. Prove that for $m \in \mathbb{Z}$, $n \in \mathbb{N}$, $(y^{1/n})^m = (y^m)^{1/n}$.

iii. Let $r \in \mathbb{Q}$, and assume that one can write $r = \frac{p}{q} = \frac{m}{n}$ for $p, m \in \mathbb{Z}$ and $n, q \in \mathbb{N}$. Prove that $(y^{1/q})^p = (y^{1/n})^m$. Define $y^r = (y^{1/q})^p$. Explain why y^r is well-defined.

iv. The last part defines a number which is formally the r-th power of y. In this last part you show it satisfies the power properties.
Prove that for $r_1, r_2 \in \mathbb{Q}$, $y^{r_1+r_2} = y^{r_1}y^{r_2}$ and $(y^{r_1})^{r_2} = y^{r_1 r_2}$.

Solutions

1. There are two differences from Midterm I, Problem 4(d). One is that the sequence may convergex to $+\infty$. In this case the claim is trivial because (see note below the problem) any real number is $\leq \infty$, after extending the $\leq$ relation to extended numbers. So this has been taken care of. Another difference is that we only assume that (a_n) is nonnegative from some index (and not from the first index). Does this make a difference ? No. Why ? Limit does not change if the sequence is changed on a finite number of indices. So modify the sequence whenever is negative (if at all) and the limit will not change. Prove it !

   (a) If $y_n \to \infty$, as $n \to \infty$, then the claim is trivial (see note below the problem). Assume then that $\lim_{n\to\infty} y_n < \infty$. Now let $c_n = y_n - x_n$, then (c_n) satisfies the conditions of the first part, and its limit is nonnegative. In particular, by the allowed arithmetic operations with extended limits, it follows that $lim_{n\to\infty} c_n = \lim_{n\to\infty} y_n - \lim_{n\to\infty} x_n$ (make sure you understand why this is allowed: not both limits are infinite), and the claim follows.

   (b) Obviously, if $y_n \to \infty$ as $n \to \infty$ then $x_n \to \infty$ as $n \to \infty$ (make sure you understand the analogous statement when the limit is $-\infty$). Therefore, we may assume that the limit of (y_n) and (x_n) is finite. Denote it by L. Then $|y_n - L| = |x_n - L + y_n - x_n| \leq |x_n - L| + |y_n - x_n| \leq |x_n - L| + z_n - x_n$. Take the limit on the right-hand side and you're done.

2. First, we show that (a_n) is unbounded above. Indeed, $a_1 \leq a_n$ for all n because the sequence is non-decreasing, so a_1 is a lower bound. Since the sequence is unbounded, it follows that it has no upper bound. In particular, for each $M \in \mathbb{R}$ there exists $N(M) \in \mathbb{N}$ such that $a_N \geq M$. But for all $n \geq N(M)$, $a_n \geq a_N$, proving, by definition, that $a_n \to \infty$ as $n \to \infty$.

3. (a) $a_1 > 0$, so by induction $a_n > 0$. Now note that $a_{n+1} = \frac{a_n}{1+a_n}$. Therefore $a_{n+1}/a_n = \frac{a_n}{a_n+1} * \frac{1}{a_n} = \frac{1}{1+a_n} < 1$. So (a_n) is non-increasing. Since it's bounded below, it converges to a limit, say L. In particular, $L = \lim_{n\to\infty} a_{n+1} = \lim n \to \infty \frac{a_n}{1+a_n} = \frac{L}{1+L}$. Solving the equation we find that $L = 0$.

   (b) (a_n) is strictly increasing. Therefore converges in the extended sense (either finite or infinite limit). Note that $a_{2n} - a_n = \frac{1}{n+1} + \cdots + \frac{1}{2n} \geq \frac{n}{2n} = \frac{1}{2}$. Therefore (a_n) is not Cauchy, so does not converge (to a finite limit), and we conclude that $\lim_{n\to\infty} a_n = \infty$.

   (c) $a_{n+2} = a_n + (-1)^n(n+1) + (-1)^{n+1}(n+2) = a_n + (-1)^n((n+1) - (n+2)) = a_n - (-1)^n$. Thus, for all odd indices n, $a_{n+2} = a_n + 1$, while for even indices $a_{n+2} = a_n - 1$. Therefore (a_n) has a subsequence converging to $+\infty$ and a a subsequence converging to $-\infty$. In particular, it does not converge, not even in the extended sense.

4. This is an exercise in rewriting the expressions. Indeed,

$$a_k = \frac{p_n k^n(1 + \frac{p_{n-1}}{p_n}k^{-1} + \ldots \frac{p_0}{p_n}k^{-n})}{q_m k^m(1 + \frac{q_{m-1}}{p_m}\frac{1}{k} + \ldots \frac{p_0}{p_m}k^{-m})} = \frac{p_n}{q_m}k^{n-m}.$$

   Note that $\lim_{k\to\infty} b_k = 1$, and the result follows from the arithmetic rules, and I'll skip the details.

5. We'll show that the Cauchy condition fails with $\epsilon = 1$. Here's why. Choose any $N \in \mathbb{N}$. We will find that there exist $n, m \geq N$ such that $|a_n - a_m| > \epsilon = 1$. Indeed, let $n = N$ and since $a_n \to \infty$ as $n \to \infty$, there exists some $m > n$ for which $a_m > a_n + 1$, or $|a_m - a_n| > 1$.

6. No. Let $a_n = n$. Then (b_n) is a constant sequence, but (a_n) is not Cauchy.

7. Let $b_n = \sup\{|a_k - a_n| : k > n\}$. Note that (b_n) is non-increasing.
   $\Rightarrow$ (assuming (a_n) is Cauchy). Fix $\epsilon > 0$, so for (any) $k \geq N(\epsilon)$, $|a_k - a_N| < \epsilon$. In particular, $b_N < \epsilon$. Since (b_n) is nonnegative and non-increasing, it follows that $|b_n - 0| = b_n < \epsilon$ for all $n \geq N(\epsilon)$. Hence $\lim_{n\to\infty} b_n = 0$.
   $\Leftarrow$ Fix ϵ. Let $N(\epsilon)$ be such that $b_n \leq \epsilon$ for all $n \geq N$. In particular, if $m \geq n \geq N$, we have $|a_m - a_n| \leq b_n < \epsilon$.

8.

   - E may be empty, for example if (a_n) converges to ∞. Assume E is nonempty, and let (x_n) be any convergent sequence of elements in E (the convergence is in $\mathbb{R}$). We need to show that the limit is in E, that is, it is a limit of some subsequence as well. Here's what we do. For each l, let $(a_{n_k^{(l)}})_{k=1}^{\infty}$ be a subsequence converging to x_l, and assume that $\lim_{l\to\infty} x_l = x$. Construct a new subsequence $(a_{\tilde{n}_k})$ by letting $n_1 = n_1^{(1)}$, and continue inductively as follows:

$$\tilde{n}_{l+1} = \min\{n_m^{(l+1)} > \tilde{n}_l : |a_{n_m^{(l+1)}} - x_{l+1}| \leq 1/l\}.$$

     This provides a subsequence because for each fixed l, $a_{n_m^{(l)}} \to x_l$ as $m \to \infty$. Now by the triangle inequality, $|a_{\tilde{n}_l} - x| \leq \frac{1}{l} + |x - x_l| \to 0$ as $l \to \infty$, completing the proof.

- Here's how we're going to proceed. We will construct a sequence which will be obtained through concatenation of finite lists. Each list will contain some approximation to the elements of E. Let's begin.

  (a) The construction.
  Let $E_n = E \cap [-n, n]$. Note that $E = \cup_{n=1}^{\infty} E_n$. Recall that for each $x \in \mathbb{R}$ there exists a unique infinite decimal representation. By infinite we mean that the representation does not terminate (with 0's). For concreteness, the number 10 has an infinite representation as $9.99999\ldots$, the number 1234.432 has an infinite representation as 1234.4319999, etc. . For any $x \in \mathbb{R}$, define its n-truncation as the number resulting by truncating its decimal expansion after the n'th decimal digit. For example, the 1-truncation of $0.234 = 0.233999$ is 0.2, and its 4-truncation is 0.2339 (note we only use the infinite representation for x, not for its truncations). Good. Note that the set of all n-truncations of numbers in $[-n, n]$ is finite (you can easily show that there are exactly $(2n + 1)10^n$). Let A_n denote the set of all n-truncations of elements in E_n. List them in increasing order, $a_{n,1}, a_{n,2}, \ldots, a_{n,L(n)}$, where $L(n)$ is the number of elements in A_n. Now let (a_n) be the sequence obtained by concatenating these finite sequences, writing all (finite) elements of A_1 first, then all elements of A_2, etc. An explicit formula for a_n is simple but not very helpful here. We're done with the construction.

  (b) The properties.
  Now let's test it. By construction, any element in E is a limit of some subsequence of (a_n), because one can choose the subsequence of its truncations. Conversely, we want to show that if x is a limit of some subsequence (a_{n_k}), then $x \in E$. Without loss of generality (and by possibly passing to a subsequence), there is no loss of generality assuming $a_{n_k} \in E_{n_k}$. Now let x_{n_k} be any element in E whose n_k-truncation is a_{n_k}. Then $|x_{n_k} - x_{n_l}| \leq |x_{n_k} - a_{n_k} + a_{n_k} - a_{n_l} + a_{n_l} - x_{n_l}| \leq |x_{n_k} - a_{n_k}| + |a_{n_k} - a_{n_l}| + |a_{n_l} - x_{n_l}|$. Now note that by definition, the distance between any number and its n-truncation is no-greater than 10^{-n}. Hence the sum of the last three terms is bounded above by $10^{-l} + 10^{-k} + |a_{n_k} - a_{n_l}| \to 0$ as $k, l \to \infty$. Therefore (x_{n_k}) is a Cauchy sequence in E and its limit coincides with the limit of (a_{n_k}), x. Since E is closed, $x \in E$.

9. This problem is straightforward, yet tedious and long. I omit a solution.

HW 6 (Posted Week 7, 10/15)

1. True or false: A sequence (a_n) converges if and only if any subsequence has a convergent subsequence (**corrected 10/26 3:00PM**)
2. Find an example of a sequence (a_n) such that $\limsup a_n = 0$ but $a_n < 0$ for all n.
3. Prove that a sequence (a_n) is bounded above if and only if $\limsup_{n\to\infty} a_n < \infty$.
4. Prove $\liminf_{n\to\infty} a_n = -\limsup_{n\to\infty}(-a_n)$.
5. Prove that $\limsup_{n\to\infty}(a_n + b_n) \leq \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} a_n$ and find an example for a strict inequality. Formulate and prove the corresponding statement for $\liminf$.
6. Find $\liminf, \limsup$ and $\lim$ if exists of the following:
   (a) $a_n = \frac{5}{n} + (-1)^n$
   (b) $a_n = \frac{n^2+3n+(-1)^n(n-1)(n+2)}{n+1}$
   (c) (a_n) any sequence containing all positive integer powers of $\frac{1}{2}$, each power appearing only once (**Corrected 12/02 8:48AM).**
7. (*) Fekete's Lemma.
   Let (a_n) be a sequence of positive numbers satisfying $a_{n+m} \leq a_n + a_m$. Prove that $\lim_{n\to\infty} \frac{a_n}{n} = \liminf_{n\to\infty} \frac{a_n}{n} = \inf\{\frac{a_n}{n} : n \in \mathbb{N}\}$
8. Prove that $\lim_{x\to x_0} f(x)$ exists if and only if the left and right limit of f at x_0 exist and are equal.
9. (a) Let $f : \mathbb{R} \to \mathbb{R}$ be defined through

   $$f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & \text{otherwise.} \end{cases}$$

   Show that for all x, f has no limit at x.

   (b) Let $f : \mathbb{R} \to \mathbb{R}$ defined through

   $$f(x) = \begin{cases} \frac{1}{q} & \text{if } x = \frac{p}{q} \text{ relatively prime } p \in \mathbb{Z},\ q \in \mathbb{N} \\ 0 & \text{otherwise} \end{cases}$$

   Find all points where f has a limit.
10. Let P and Q be polynomials. Let $R(x) = \frac{P(x)}{Q(x)}$, defined for all x such that $Q(x) \neq 0$. Using the arithmetic properties of limits, prove that R is continuous on its domain.

Solutions

1. False. The direct implication is true, however the reverse is false. Any bounded sequence satisfies the condition, but not every bounded sequence converges. This of course is just the principle. A concrete example is needed: let $a_n = (-1)^n$. Satisfies the condition because it's a bounded sequence but does not converge.

2. $a_n = -\frac{1}{n} - (1 + (-1)^n)n$. $\limsup_{n\to\infty} a_n = 0$, but $a_n \leq -\frac{1}{n}$. Note $\liminf_{n\to\infty} a_n = -\infty$ (I chose this example because its a divergent sequence).

3. $\Rightarrow$ Bounded above, then $a_n \leq M$ for some finite M. Therefore $\limsup_{n\to\infty} \leq \sup\{a_k : k \geq n\} \leq M < \infty$.
$\Leftarrow$. Suppose $\limsup_{n\to\infty} a_n = M < \infty$. Then for some N, $\sup\{a_k : k \geq N\} \leq M + 1$ and so for all n: $a_n \leq M + 1 + \max\{|a_k| : k \leq N\} < \infty$.

4. $$\liminf_{n\to\infty} a_n = \lim_{n\to\infty} \inf\{a_k : k \geq n\} = -\lim_{n\to\infty}(-\inf\{a_k : k \geq n\}) = -\lim_{n\to\infty} \sup\{-a_k : k \geq n\} = -\limsup_{n\to\infty}(-a_n).$$

5. Let (n_k) be a subsequence such that $\lim_{k\to\infty}(a_{n_k} + b_{n_k}) = \limsup_{n\to\infty}(a_n + b_n)$. Fix $\epsilon > 0$. Let K be such that for $k \geq K$, $a_{n_k} \leq \limsup_{n\to\infty} a_n + \epsilon/2$, $b_{n_k} \leq \limsup_{k\to\infty} b_n + \epsilon/2$. Then for those k,
$$a_{n_k} + b_{n_k} \leq \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n + \epsilon.$$
The left-hand side converges to $\limsup_{n\to\infty}(a_n + b_n)$ and ϵ is arbitrary. Take $a_n = (-1)^n$ and $b_n = -(-1)^n$. Then for all n, $a_n + b_n = 0$ while $\limsup_{n\to\infty} a_n = \limsup_{n\to\infty} b_n = 1$.

6. (a) $\limsup_{n\to\infty} a_n = 1$, $\liminf_{n\to\infty} a_n = -1$.
   (b) Split to even and odd.
   Even n: $a_n = \frac{2n^2+4n-2}{n+1} \to \infty$ as $n \to \infty$.
   Odd n: $a_n = \frac{2n-2}{n+1} \to 2$ as $n \to \infty$.
   Thus $\limsup = \infty$, $\liminf = 2$.
   (c) Let $\epsilon > 0$. Let $N(\epsilon)$ be such that for all $n \geq N(\epsilon)$, $a_n < \epsilon$. This is possible because there are finitely many elements in the sequence bigger or equal to ϵ. Let $m \geq N$ as well. Then $|a_n - a_m| \leq \max\{a_n, a_m\} < \epsilon$.
   Summary: $\lim_{n\to\infty} a_n = 0$.

7. There is no loss of generality defining $a_0 = 0$, as it does not alter the hypothesis. First some observations. Note that by the assumption we have $a_{n+m} \leq a_n + a_m$ and by simple induction, $a_{km} \leq ka_m$ and so $a_{km+l} \leq ka_m + a_l$. Clearly $a_n \leq na_1$, so the sequence $(a_n/n)_{n=1}^{\infty}$ is bounded (it's bounded from below because the numbers are nonnegative). Also note that for all $n \in \mathbb{N}$, $\frac{a_{2n}}{2n} \leq \frac{2a_n}{2n} = \frac{a_n}{n}$, so $\inf\{a_n : n \in \mathbb{N}\} = \liminf_{n\to\infty} a_n$.
Now the proof. Fix some $\epsilon > 0$. Let $A = \liminf_{n\to\infty} \frac{a_n}{n}$. Let m be such that $a_m/m \leq A + \epsilon$. Any n could be written as $n = km + l$ where k is a nonnegative integer and $l \in \{0, \ldots, m-1\}$ (k is the integer part of n/m and l is the remainder when dividing n by k or $l = n \mod k$). Now we have
$$\frac{a_n}{n} = \frac{a_{km+l}}{km+l} \leq \frac{ka_m + a_l}{km+l} \leq \frac{ka_m}{km} + \frac{a_l}{km} \leq A + \epsilon + \frac{\max\{a_j : j < m\}}{k}.$$
Therefore when taking $n \to \infty$, also $k \to \infty$ and the right-hand side goes to $A + \epsilon$. Since $\epsilon > 0$ is arbitrary, this implies $\limsup \frac{a_n}{n} \leq A = \liminf_{n\to\infty} \frac{a_n}{n}$, proving convergence.
Beautiful, right ?

8. $\Rightarrow$ Trivial.
$\Leftarrow$ Let L denote the common limit. Fix $\epsilon > 0$. Then there exist $\delta_+(\epsilon), \delta_-(\epsilon) > 0$, such that for if $x \in E$ with $0 < x - x_0 < \delta_+(\epsilon)$ then $|f(x) - L| \leq \epsilon$ and if $x \in E$, with $0 < x_0 - x < \delta_-(\epsilon)$, $|f(x) - L| \leq \epsilon$. Set $\delta(\epsilon) = \max(\delta_+, \delta_-)$m and the result follows.

9. (a) Both rationals and irrationals are dense in $\mathbb{R}$. In particular for every $x_0 \in \mathbb{R}$ there exists a sequence of rationals converging to x_0 and a sequence of irrationals converging to it. The limit of f along the first is 1 while the limit along the other is 0. Thus, limit (along all sequences) does not exist.
   (b) The function f has a limit at all points. However it continuous only irrational points and 0. Let's prove that.
   Indeed, let $x_0 \in \mathbb{R}$ and let (a_n) be any sequence converging to x_0. If (a_{n_k}) is a subsequence of irrationals converging to x_0, then for each $m \in \mathbb{N}$ there exists some $l \in \mathbb{Z}$ such that for all k sufficiently large $a_{n_k}, x_0 \in [l/m!, (l+1)/m!)$. This interval does not contain any rationals in reduced form with denominator less than or equal to m, except, possibly for the number 0 (why ?). Therefore, the denominator of a_{n_k} in reduced form is bounded below by m. Since m is arbitrary, this proved that the denominator of a_{n_k} converges to ∞ as $k \to \infty$. Consequently $f(a_{n_k}) \to 0$ as $k \to \infty$. Finally for all n such that a_n is irrational, $f(a_n) = 0$. Therefore $f(a_n) \to 0$ as $n \to \infty$. This proves that the limit exists and is equal to 0. Since $f(x_0) = 0$ if and only if x_0 is irrational or $x_0 = 0$, it follows that f is continuous on the irrationals and 0.

10. Use sequences.

HW 7 (Posted Week 8, 10/22)

Section 3.2:9,10
Section 3.3:3,4,5,7,8,10
Section 3.4:1,4,5

Solutions

3.2:9 Assume $f(q) = 0$ for all rational $q \in [0,1]$. Then since for each $a \in [0,1]$ there exists a sequence of rationals converging to a, it follows that $f(a) = 0$. The converse is trivial.

3.2:10 Let $a_n = f(n) - f(n-1)$. Then $a_n \to L$. Fix $\epsilon > 0$ and let N be such that $|a_n - L| \leq \epsilon$ for $n \geq N$. Then for all $n > N$ we have $f(n) = f(N-1) + a_N + \cdots + a_n$. Therefore

$$|\frac{f(n)}{n} - L| = |\frac{f(N-1)}{n} + \frac{(a_N - L) + \ldots (a_n - L)}{n} + \frac{((n-N+1)-n)}{n} L| \leq \frac{|f(n-1)|}{n} + \frac{n-N+1}{n}\epsilon + \frac{(N+1)|L|}{n} \leq 2\epsilon,$$

if n is sufficiently large. This completes the proof.

3.3:3 This is the extreme value theorem, because $|\cdot|$ is continuous and so $|f|$ is a composition of continuous functions, hence continuos.

3.3:4 Let $\epsilon = (M - f(a))/2$. Then by continuity, there exists some $\delta(\epsilon) > 0$ such that whenever $|y - a| < \delta$, $|f(y) - f(a)| \leq \epsilon$. In particular $f(y) \leq f(a) + \epsilon < f(a)/2 + M/2 < M$.

3.3:5 Let $f(x) = 1$ if $x \in \mathbb{Q}$ and 0 otherwise, and let $g = 1 - f$. This works both for sum and product: $f + g = 1$ and $fg = 0$.

3.3:7 (a) $g = (f+g) - f$, therefore it follows from arithmetic. (b) Similarly, $g = (fg)/f$. Therefore g is continuous if fg is continuous and $f(a) \neq 0$.

3.3:8 (a) $f(2x) = f(x) + f(x)$ and continue by induction to show that $f(nx) = nf(x)$ for all $x \in \mathbb{R}$ and $n \in \mathbb{N}$. In particular $f(0) = f(2 \cdot 0) = f(0) + f(0)$. Therefore $f(0) = 0$. Thus, $0 = f(0) = f(x + (-x)) = f(x) + f(-x)$. Therefore $f(-x) = -f(x)$. In particular, for $n \in \mathbb{N}$, $f(-nx) = nf(-x) = -nf(x)$.
(b) Write $q = n/m, n \in \mathbb{Z}, m \in \mathbb{N}$. Then $mf(qx) = f(mqx) = f(nx) = nf(x)$. Therefore $f(qx) = n/mf(x) = qf(x)$. In particular $f(q) = qf(1)$.
(c) $|f(x) - f(a)| = |f(x - a + a) - f(a - a + a)| = |f(x-a) - f(0)|$. Therefore f is continuous at 0 if and only f is uniformly continuous.
(d) By part (b) $f(x) = \lim_{q \in \mathbb{Q}, q \to x} qf(1) = xf(1)$. Therefore $f(x) = xf(1)$. Clearly, $m = f(1)$.

3.3:10 Let $m = \inf\{f(x) : x \in \mathbb{R}\}$. Let (x_n) be a sequence such that $f(x_n) \to m$ as $n \to \infty$. By assumption, there exists some $N > 0$ such that for all $|x| > N$, $f(x) > m + 1$. Therefore, (x_n) is bounded. In particular, it has a convergent subsequence, and the result follows from continuity.

3.4:1 (a) $|x^3 - y^3| = |x - y||x^2 + xy + y^2| < |x - y|3$. Therefore letting $\delta = \epsilon/3$ does the work. (b) $|x^2 - x - (y^2 - y)| \leq |x^2 - y^2| + |x - y| = |x - y||x + y| + |x - y| \leq 3|x - y|$. Same δ as above. (c) $|x\sin(2x) - y\sin(2y)| = |(x - y)\sin 2x + y(\sin 2x - \sin 2y)|$. Use the fact that $|\sin\alpha - \sin\beta| \leq |\alpha - \beta|$ to conclude with $\delta = \epsilon/5$.

3.4:5 Fix ϵ. There exists $M > 0$ such that for $x, y \geq M$, $|f(x) - f(y)| \leq \epsilon$. Since f is continuous on $[0, M]$, there exists $\delta > 0$ such that if $x, y \in [0, M]$ and $|x - y| < \delta$ then $|f(x) - f(y)|\epsilon$. Finally, if $x < M$ and $y > M$, then $|f(x) - f(y)| \leq |f(x) - f(M)| + |f(M) - f(y)| \leq 2\epsilon$. Thus for all $|x - y| < \delta$, $|f(x) - f(y)| < 2\epsilon$.

HW 8 (Posted Week 9,10/29)

Section 4.1:2,3,4,5,6,7
Section 4.2:2,7,8

1. (*) A function f is non-decreasing if whenever $x \leq y$, $f(x) \leq f(y)$. Prove that a non-decreasing function is continuous, possibly except a countable number of points.
2. Assume that the limit $\lim_{h\to 0} \frac{f(x+h)-f(x-h)}{2h}$ exists for all x.
   (a) Assume that f is differentiable. Identify the limit.
   (b) Does the limit imply that f is differentiable ? Continuous ?

Solutions

4.1:2 We work under the assumption that for $\beta > 0$, the function $|x|^\beta$ is continuos, and its value at $x = 0$ is 0. Clearly $f(0) = 0$ (squeeze). Therefore $|\frac{f(x)-f(0)}{x-0}| \leq |x|^{\alpha-1} \to 0$ as $x \to 0$. Hence the derivative exists and is equal to 0. The function $|x|$ is not differentiable at 0. Left derivative -1 and right derivative $+1$.

4.1:3 (a) This is obvious from the definition (b) Derivative when computing limit from the left is negative and when computing from the right is positive. Therefore limit must be zero. (c) Identical, with opposite signs. (d) $f(x) = x^3$

4.1:4 Note that $|\sin x| \leq |x|$. Therefore sin continuous at 0. Since $\cos x = 1 - 2\sin^2(x/2)$ it follows that cos is continuous at 0. Now $\sin(x+h) = \sin x \cos h + \cos x \sin h$. Thus, $\lim_{h\to 0} \sin(x+h) = \sin x \cos 0 + \cos x \sin h = \sin x$ (we've used the continuity of $\sin, \cos$ at 0). Therefore sin is continuous everywhere. Same half-angle identity implies cos is continuous everywhere. We only do the first limit. Second follows from half angle identity. Since sin and x are anti-symmetric, it is sufficient to consider $x > 0$. Now $\sin x/x \leq 1$, but also $\sin x/x \geq \cos x \to 1$ as $x \to 0$. This completes the proof. In particular, sin is differentiable at 0. Consequently, cos is differentiable at 0. Finally, use the formula for $\sin(x+h)$ to conclude that sin is differentiable at any x (it reduces to differentiability at 0). You know what to do for cos. Write the explicit expression to finish up.

4.1:5 $\lim_{x\to y} f(x) - f(y) = \lim_{x\to y} f(x/y) = \lim_{h\to 1} f(h)$, if at least one of the limits exists. Therefore f is continuous at y if the limit exist and is equal to 0, which is equivalent to the statement that f is continuous at 0. As for differentiablity $\frac{f(x)-f(y)}{x-y} = \frac{f(x/y)-f(1)}{y(x/y-1)}$. Therefore f is differentiable at y if and only if it is differentiable at 0 and then $f'(y) = \frac{f'(1)}{y}$.

4.1:6 (a) We did this in class. (b) Use something a little more advanced. The function $f(x) = x^n$ is strictly increasing and its inverse is $g(y) = y^{1/n}$. Therefore, apply the theorem on derivative of inverse function to conclude.

4.1:7 Continuity is by squeeze (see comment on [4.1:2] above. Differentiate by definition an apply squeeze to conclude that the derivative is 0.

4.2:2 Too easy, I hope.

4.2:7 Use induction, starting with $n = 2$.

4.2:8 Conclude this from [4.1:6], and the composition $(x^{1/n})^m$.

1. It's in the book.
2. (a) If differentiable then $\frac{f(x+h)-f(x)+(f(x)-f(x-h))}{2h} = \frac{1}{2h}(f(x+h)-f(x)) + \frac{1}{-2h}(f(x-h)-f(x))$ so the limit is $f'(x)$.
   (b) No. Take $f(x) = |x|$ with $x = 0$. Then $f(h) - f(-h) = 0$ so limit exists and is equal to 0, but f is not differentiable at 0. Even worse, take same function and change it at 0 to be equal to 1. It's not even continuous. Still, the limit exists.

HW 9 (Posted Week 10, 11/06)

Section 4.3:5-12
Section 4.4:1(a)(b),2,5a-c,6,7,9*,10*.
Also: Suppose f is differentiable everywhere. Show that if f' has no jump discontinuities.

Solutions

4.3:5 (a) By Mean Value Theorem, $f(x) - f(0) = f'(c)(x-0) = 0$. (b) $|f(x)| - |f(0)| \le |f(x) - f(0)| = |f'(c)||x| \le |x|$. Therefore, $|f(x)| \le 1 + |x|$.

4.3:6 Midterm II problem 5(b)

4.3:7 $0 < f(c) - f(a) = f'(x_2)(c-a)$, thus $f'(x_2) > 0$. Similarly, $0 > f(b) - f(c) = f'(x_1)(b-c)$ and so $f'(x_1) < 0$.

4.3:8 $f(x_3) - f(x_2) = f'(c_2)(x_3 - x_2)$ so $f'(c_2) > 0$. Similarly, $f(x_2) - f(x_1) = f'(c_1)(x_2 - x - 1)$ so $f'(c_1) < 0$. Note $c_1 < x_2 < c_2$. Therefore $f'(c_2) - f'(c_1) = f''(c)(c_2 - c_1)$ hence $f''(c) > 0$.

4.3:9 $\frac{f(2n)-f(n)}{n} = f'(c_n)$, for some $c_n \in (n, 2n)$. The limit on the left-hand side as $n \to \infty$ is 0 and is equal to the limit on the right hand side, which coincides with L.

4.3:10 Use the hint.

4.3:11 $f(\sup E) \ge f(x)$ for all $x \in E$, and the inequality is preserved when taking sup on the right hand side. But there exists a sequence (x_n) in E converging to $\sup E$ and by continuity $f(\sup E) = \lim_{n\to\infty} f(x_n) \le \sup_{x\in E} f(x)$.

4.3:12 (a) Since f has a local maximum at x_0 and is differentiable, $f'(x_0) = 0$. Since its a proper maximum, for each $\delta > 0$, one can find $a < x_0 < b$ such that $b - a < \delta$ and so $f(a) < f(x_0)$ and $f(b) < f(x_0)$. Apply the mean value theorem to the intervals $[a, x_0]$ to obtain x_1 and $[x_0, b]$ to obtain x_2. (b) $-f$ has a proper local maximum.

4.4:1 (a) $f'(x) = 3x^2 + 2ax + 3$. Thus $f'(1) = 3 + 2a + 3 = 6 + 2a > 0$ if and only if $a > -3$. (b) $f'(x) = 2ax + 3$. Now $f(1) = 2a + 3$ and $f(2) = 4a + 3$. In order to be strictly increasing we want the derivative to be positive on $(1, 2)$ which is equivalent to the requirement $\min(2a + 3, 4a + 3) \ge 0$, or $\min(2a, 4a) \ge -3$. This obviously holds when $a \ge 0$. Otherwise, we need $4a \ge -3$ or $a \ge -3/4$.

4.4:2 (a) $(f^{-1})'(y) = \frac{1}{f'(f^{-1}(y))} = \frac{1}{f'(0)} = \pi^{-1}$. (b) Same idea, answer is e^{-1}. (c) Use product rule for derivatives to show this is equal to $\pi^{-1}1 + e^{-1}0 = \pi^{-1}$.

4.4:5
   (a) Observe that either $f'(x) > 0$ for all $x \in (a, b)$ or $f'(x) < 0$ for all $x \in (a, b)$. Indeed, if there exist x_1 and x_2 such that $f'(x_1) < 0 < f'(x_2)$, then by Intermediate Value Theorem $f'(x_0) = 0$ for some x_0, which contradicts the assumption. Without loss of generality, assume $f'(x) > 0$. Now by Mean Value Theorem for all $y > x$, $f(y) - f(x) = f'(c)(y - x) > 0$, so f is 1-1. Let $d = \sup_{x\in(a,b)} f(x) = \lim_{x\to b} f(x)$ and let $c = \inf_{x\in(a,b)} f(x) = \lim_{x\to a} f(x)$. Then by Intermediate Value Theorem f attains all values in (c, d). It does not attain c or d, because it is strictly increasing. Indeed if $f(x_0) = c$ for some $x \in (a, b)$, then the definition of c and the monotonicity of f imply $f(x) = c$ for all $x \in (a, x_0)$, violating the assumption that $f'(x) > 0$.

   (b) $(f^{-1})'(y) = \frac{1}{f'(f^{-1}(y))}$. The denominator is a composition of continuous functions and is then continuous. Since $f'(x) \ne 0$, the denominator does not vanish and hence the function on the left-hand side is continuous.

   (c) $f^{-1} = x^{1/3}$ and for $x \ne 0$, $(f^{-1})'(x) = \frac{1}{3}x^{-2/3}$, which has a type 2 discontinuity at 0.

4.4:6 Restated, we are interesting in functions such that $f'(x) = \frac{\alpha}{f'(x)}$, or $(f'(x))^2 = \alpha$. Thus, either $f'(x) = \sqrt{\alpha}$ for all x or $f'(x) = -\sqrt{\alpha}$ for all x. So all possible solutions are $f(x) = \pm\sqrt{\alpha}x + C$.

4.4:7 (a) Use problem 4.4:5. (b) By continuity there exists some open interval I around x_0 where f' does not vanish. The result follows from problem 5(a)(b).

4.4:9 If not strictly monotone, one can find $x_1 < x_2 < x_3$ such that $f'(x_1) < f'(x_2)$ but $f'(x_2) > f'(x_3)$ (or with inequalities reversed). By Intermediate Value Theorem for the derivative this implies that there exist $x_1 < y_1 <_2< y_2 < x_3$ such that $f'(y_1) = f'(y_2)$, violating the assumption.

4.4:10 On one hand a monotone function may have only jump discontinuities., while on the other hand, f' cannot have such discontinuities because of the Intermediate Value Theorem for the derivatives.

Last problem follows directly from the Intermediate Value Theorem for derivatives.

Midterm II (Week 11, 11/12)

1. (a) Define the term: f is a continuous function on $\mathbb{R}$.
   (b) State the Intermediate Value Theorem (NOT Mean Value Theorem).

2. (Limits of Sequences) Prove that $\limsup_{n\to\infty}(a_n+b_n) \leq \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n$, whenever right-hand side is defined.

3. (Continuity) Consider the function

$$f(x) = \begin{cases} \frac{1}{q} & x = \frac{p}{q}, p \in \mathbb{Z} \text{ and } q \in \mathbb{N} \text{ are relatively prime;} \\ 0 & \text{otherwise.} \end{cases}$$

   Prove that f is continuous at x_0 if and only if $x_0 = 0$ or x_0 is irrational.

4. (Differentiability) Use the definition of differentiability to prove that if f is differentiable on $\mathbb{R}$ and $f(x_0) \neq 0$, then $g(x) = \frac{1}{f(x)}$ is differentiable at x_0. Express the derivative in terms of $f(x_0)$ and $f'(x_0)$.

5. (a) (Intermediate Value) Suppose that $f : [0,1] \to \mathbb{R}$ is a continuous function satisfying $f(x) \in [0,1]$ for all $x \in [0,1]$. Prove that there exists some $x_0 \in [0,1]$ such that $f(x_0) = x_0$.
   (b) (Mean Value) Prove that if f is differentiable and its derivative is bounded (from above and below) then f is uniformly continuous.

6. Determine whether the each of the following is true or false. If true, prove. If false, find a counterexample.

   (a) If f is uniformly continuous on $\mathbb{R}$ then f is bounded.
   (b) If f is uniformly continuous on $(0,1)$, then f is bounded on $(0,1)$.
   (c) If f is continuous on $\mathbb{R}$ and $f(x) \in \mathbb{Q}$ for all x, then f is a constant function.
   (d) If $\lim_{n\to\infty}(a_{n+1} - a_n) = 0$ then (a_n) is Cauchy.
   (e) If (a_n) is non-increasing then $\limsup_{n\to\infty} a_n = \inf\{a_n : n \in \mathbb{N}\}$.
   (f) If (a_n) is a sequence containing all positive powers of $\frac{1}{3}$, each power appearing only once, then (a_n) converges to 0.

BONUS. Let (a_n) be a bounded sequence of nonnegative numbers. Suppose that $\limsup_{n\to\infty}(a_n + \frac{1}{1+a_n}) = \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} \frac{1}{1+a_n}$. Prove that (a_n) converges.

Solutions

1. Book, notebook, etc.

2. HW6, problem 5.

3. HW6, problem 9(b).

4. $\frac{\frac{1}{f(x_0+h)}-\frac{1}{f(x_0)}}{h} = \frac{f(x_0)-f(x_0+h)}{f(x_0+h)f(x_0)h} \to -\frac{f'(x_0)}{f^2(x_0)}$.

5.
   (a) Let $g(x) = f(x) - x$. Then $g(0) \geq 0$ and $g(1) \leq 0$. Therefore by the Intermediate Value Theorem, $g(x_0) = 0$ for some x_0.

   (b) Let $M = \sup_x |f'(x)|+1$. By Mean Value Theorem $|f(y)-f(x)| = |f'(u)||y-x|$ for some u between y, x. Therefore $|f(y)-f(x)| \leq M|y-x|$. Now choose $\delta = \epsilon/M$ and then whenever $|y-x| < \delta$, $|f(y)-f(x)| < \epsilon$.

6.
   (a) FALSE. $f(x) = x$.

   (b) TRUE. By the Extension Theorem f could be extended to a continuous function on $[0, 1]$. The latter is bounded, so f is.

   (c) TRUE. If f attains two distinct values q_0 and q_1, then by the Intermediate Value Theorem it must attain all values between them. Those include irrationals, and this violates the assumption.

   (d) FALSE. $a_n = \sqrt{n}$. Indeed $\lim_{n\to\infty} a_n = \infty$, yet $a_{n+1} - a_n \to 0$ (check).

   (e) TRUE, because (a_n) converges in the extended sense. In particular, $\limsup_{n\to\infty} a_n = \liminf_{n\not\infty} a_n = \inf_n a_n$.

   (f) TRUE. HW6 problem 6(c).

BONUS. Take a subsequence (a_{n_k}) attaining the $\limsup$ on the left-hand side. Without loss of generality, we may assume (by further extracting sub-subsequences) that also (a_{n_k}), and converges. This implies $\frac{1}{1+a_{n_k}}$ converges. Each of these limits is bounded above by the corresponding $\limsup$ for the full sequence (terms on the right-hand side). Thus the sum of these limits is bounded above by the right-hand side. Yet it is equal to the left-hand side, which it turn implies it is equal to the right-hand side. It follows that $\lim_{k\to\infty} a_{n_k} = \limsup_{n\to\infty} a_n$, $\lim_{k\to\infty} \frac{1}{1+a_{n_k}} = \limsup_{n\to\infty} \frac{1}{1+a_n}$. But the last expression is equal to $\frac{1}{1+\liminf a_n}$. Therefore the limit of a_{n_k} is equal to both $\limsup_{n\to\infty} a_n$ and $\liminf_{n\to\infty} a_n$, so they are the same, proving convergence.

HW 10 (Posted Week 12, 11/18)

Submit your solutions to the boxed problems by the end of the lecture on Thursday, 12/03.

The assignment will be graded. The grade will replace Quiz 6, which was canceled.

Sec. 5.1: 3, $\boxed{4}$, $\boxed{5}$, 8

1. Let $f : [a, b] \to \mathbb{R}$. Define the positive part of f, f_+, by letting $f_+(x) = \max\{f(x), 0\}$. Similarly, define the negative part of f, f_-, by letting $f_-(x) = \max\{-f(x), 0\}$. Prove that f is integrable on $[a, b]$ if and only if f_+ and f_- is integrable on $[a, b]$ and then $\int_a^b f dx = \int_a^b f_+ dx - \int_a^b f_- dx$. (**correction, 12/01 5:11PM**)

$\boxed{2.}$ Prove by using the definition that the function $f(x) = x$ is integrable on $[a, b]$. Use this method to calculate the integral.

You may want to use the identity $1 + 2 + \cdots + (n - 1) = n(n - 1)/2$.

Solutions

5.1:3 Fix some large N. Choose a partition which includes $x_0 = 0, x_1 = \frac{1}{N}$ and $x_n = 1$, and in between every $\frac{1}{k}$, $k = 1, \ldots, N$ is contained in an interval in the partition of length $\leq \frac{1}{N^2}$. It follows that $L(f, P) = 0$ and $U(f, P) \leq \frac{1}{N} + N\frac{1}{N^2} \to 0$ as $N \to \infty$. Therefore $\int_0^1 f(x)dx = 0$.

5.1:4 (a) $|f|$ is continuous and non-negative. By assumption $f(x_0) \neq 0$ and continuity, there exists a non-degenerate interval over which $|f|(x) > \epsilon$ for $\epsilon = |f|(x_0)/2$. Taking any partition which contains this interval, we obtain $L(f, P) > 0$ for that partition and so the lower Riemann sum, the supremum of $L(f, P)$ over all partitions is > 0. (b) If $f = 0$, then $|f| = 0$ and so its integral is 0. Conversely, apply part (a).
(c) No. $\int_{-1}^1 x dx = 0 < 1 = \int_{-1}^1 |x| dx$.

5.1:5 $\int_d^e f(x)dx = \int_a^e f(x)dx - \int_a^f f(x)dx = 0$. Since right-hand side is always 0, the integral of f over any interval $[d, e] \subset [a, b]$ is 0. Assume now f is non-zero for some x_0. Then it remains such over some interval $[d, e]$, but then its integral over this interval is non-zero, a contradiction.

5.1:8 Without loss of generality, assume f is non-decreasing. Then $0 \leq (M_j - m_j)(x_j - x_{j-1}) \leq (M_j - m_j)\|P\|$. Summing over all j's, we observe $L(f, P) - U(f, P) \leq (f(b) - f(a))\|P\|$. By letting $\|P\| \leq \frac{\epsilon}{1+f(b)-f(a)}$, one integrability follows.

1. Let P be any partition. and let $M_j^{\pm}, m_j^{\pm}$ denote the sup and inf of $f_{\pm}$ on the j'th interval of the partition, respectively. Now by definition $M_j^{\pm} \leq |M_j|$ and $m_j^{\pm} \geq |m_j|$. Therefore $|M_j^{+} - m_j^{+}| \leq |M_j| - |m_j| \leq |M_j - m_j|$. Thus, f_+ is integrable if f is. Similarly, f_- is integrable if f is. Conversely, if f_+ and f_- are integrable, then $f = f_+ - f_-$ is integrable. Thus f is integrable if and only if f_+, f_- are.
2. Choose the partition $x_k = k/n$ and estimate the resulting lower, upper sums.

HW 11 (Posted Week 14, 12/03)

Section 5.2:3,5,8.
Section 5.3: 1,3,4(e)
Section 6.1:2(b)(c),4,5,6,10
Section 6.2:1,2,3

1. Complete the proof of the theorem on properties of the Riemann integral (sum of integrable functions, multiplication by a constant, monotonicity, absolute value, splitting an interval).
2. Show that the function $\ln x$ defined in class is unbounded from above and below.

Solutions

5.2:3 Let M be the supremum of f over $[0,1]$. Then $n^\alpha \int_0^{n^{-\beta}} f(x)dx \le n^\alpha n^{-\beta} M \to 0$ as $n \to \infty$. Similarly, let m denote the infimum of f over $[0,1]$ and to show that $n^\alpha \int_0^{n^{-\beta}} f(x)dx \ge n^{\alpha-\beta} m \to 0$.

5.2:5 (a) Let $M = \sup_{x\in[a,b]} |f(x)|$. Now $|fg_n|(x) \le Mg_n(x)$, so $U(fg_n, P) \le MU(g_n, P) \to 0$. But also $U(-fg_n, P) = -L(fg_n, P)$ satisfies this inequality (by replacing f with $-f$). Therefore $L(f, P) \ge -MU(g_n, P) \to 0$.
(b) Take $g_n = x^n$ and recall $\int_0^1 x^n dx = \frac{1}{n+1}$.

5.2:8 (a) The second inequality is trivial. We only prove the first. Find a nondegenerate interval where $|f|(x) \ge M - \epsilon$, which exists due to continuity of f.
(b) Take p-th root in (a) and recall that $x^{1/p} \to 1$ for all $x > 0$. Then the result follows from the squeeze.

5.3:1 That you should know.

5.3:3 (a) $2xf(x^2)$ (b) $h(t) + \sin t h(\cos t)$ (c) $\int_0^t g(x-t)dx = \int_{-t}^0 g(x)dx$. Now differentiate. The derivative is then $g'(-t)$. (d) Integrate by parts.

$$\int_0^1 x^2 f(x)dx = \frac{1}{3}f(1) - \frac{1}{3}\int_0^1 x^2 f'(x)dx = \frac{1}{3}f(1) - \int_0^1 x^5 e^{x^6}dx = \frac{1}{3}f(1) - \frac{1}{6}e^{x^6}.$$

Now complete the details.

5.3:4(e) Let $a_n = \ln(1+\frac{1}{n})^n) = \frac{\ln(1+\frac{1}{n})}{1/n}$. Let $g(x) = \ln(1+x)/x$. Then $a_n = g(1/n)$. Note that $\lim_{x\to 0} g(x) = \frac{d}{dx}\ln(1+x)|_{x=0} = 1$. Therefore $\lim_{n\to\infty}(1+\frac{1}{n})^n = \lim_{n\to\infty} e^{a_n} = e^{\lim_{n\to\infty} a_n} = e^1$, because $e^{\cdot}$ is a continuous function.

6.1:2 (b) $\sum_{k=1}^\infty (-.2)^k + 4\sum_{k=1}^\infty (.2)^k = \frac{.2}{1.2} + \frac{0.8}{0.8} = \frac{1}{6} + 1$.
(c) $7\sum_{k=1}^\infty (3/7)^k = ...$

6.2:4 For $x \neq 0$, $(x^k - x^{k-1})(x^k + x^{k-1}) = x^{2k}(1 - x^{-2})$. Therefore convergence if and only if $|x| \leq 1$.

6.1:5 (a) $a_k \to 1$ (b) $a_k \to -e$ (take logarithm) (c) $a_k/(1/k) \to 1$

6.1:6 Convergence of series is by definition convergence of the sequence of partial sums. A convergent sequence is bounded. Converse is false: $a_k = (-1)^k$. Partial sums alternate between -1 and 0.

6.1:10 Let $\epsilon > 0$. Choose N large such that $\sum_N^\infty a_k/k < \epsilon$. Then $\limsup_{j\to\infty} \sum k = 1^\infty a_k/(j+k) = \lim_{j\to\infty} \sum_{k=1}^N a_k/(j+k) + \epsilon = \epsilon$.

6.2:1 (a) Limit comparison with $\sum \frac{1}{k^2}$.
(b) Same with $\sum 2^{-k}$.
(c) Use $\ln k/k^\epsilon \to 0$ as $k \to \infty$. Therefore limit comparison with $\sum k^{-p+\epsilon}$ for $\epsilon < p-1$.
(d) Limit comparison with $\sum 3^{-k}$.
(e) Same with $\sum k^{-e}$. (f) Same with $\sum k^{-2}$.

6.2:2 (a)-(c) Limit comparison with $\sum k^{-1}$.
(d) Condensation test as done in class.

6.2:3 $p > 1$ by condensation test as done in class.