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PDE FOR APPS F 08, HW12 SOLUTION

Below is a link to a Java applet I found that graphs phase portraits.
http://www.math.psu.edu/melvin/phase/newphase.html

9.1:1 To find eigenvalues observe that

$$\det(A-\lambda I)=\det\begin{pmatrix}3-\lambda & -2\\ 2 & -2-\lambda\end{pmatrix}=(\lambda-3)(2+\lambda)+4=\lambda^2-\lambda-2=(\lambda-2)(\lambda+1).$$

Therefore $r_1=2>0>r_2=-1$ and $\binom{0}{0}$ is a saddle point and is unstable. As for eigenvalues we need to solve $Av=\lambda v$. It's clear that the vector $\binom{0}{1}$ is not an eigenvector, therefore we may assume that $v=\binom{1}{v_2}$. Letting $\lambda=r_1=2$ we see that $3-2v_2=2$ hence $v_2=\frac{1}{2}$. Letting $\lambda=r_2=-1$ we see that $3-2v_2=-1$ therefore $v_2=2$. Summarizing: $\binom{2}{1}$ is eigenvector for $r_1=2$ and $\binom{1}{2}$ is eigenvector for $r_2=-1$.

9.1:4

$$\det(A-\lambda I)=\det\begin{pmatrix}1-\lambda & -4\\ 4 & -7-\lambda\end{pmatrix}=(\lambda-1)(7+\lambda)+16=\lambda^2+6\lambda+9=(\lambda+3)(\lambda+3).$$

Therefore $r_1=r_2=-3$. As for eigenvalues we need to solve $Av=-3v$. It's clear that the vectors $\binom{0}{1}$, $\binom{1}{0}$ are not eigenvectors, therefore we may assume that $v=\binom{1}{v_2}$, $v_2\neq 0$. We see that $4-7v_2=-3v_2$ hence $v_2=1$. This shows that there's only one independent vector. We may then choose the eigenvector to be $v=\binom{1}{1}$. At this stage we are able to conclude that $\binom{0}{0}$ is an improper sink and is asymptotically stable. We seek for the generalized eigenvector, obtained by $(A+3I)u=v$. Since all solutions to this equation differ by a constant multiple of v (why ? the null space of $(A+3I)$ is spanned by v) there exists a solution of the form $u=\binom{0}{u_2}$ (one can always add a constant multiple of v to make the first component equal to 0). The equation for u_2 is $4*0-4u_2=1$. Therefore $u=\binom{0}{-\frac{1}{4}}$.

9.1:12

$$\det(A-\lambda I)=\det\begin{pmatrix}2-\lambda & -\frac{5}{2}\\ \frac{9}{5} & -1-\lambda\end{pmatrix}=\lambda^2-\lambda+\frac{5}{2}=(\lambda-\frac{1}{2})^2+\frac{9}{4}.$$

Hence $\lambda-\frac{1}{2}=\pm\frac{3i}{2}$. In particular $r_{1,2}=\frac{1}{2}(1\pm 3i)$, and we conclude that $\binom{0}{0}$ is a spiral source. Therefore it is unstable.

9.1:15 First we find the critical point by solving

$$\begin{pmatrix}-1 & -1\\ 2 & -1\end{pmatrix}v+\begin{pmatrix}-1\\ 5\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix},$$

where $v=\binom{v_1}{v_2}$. Thus $-v_1-v_2=1$ and $2v_1-v_2=-5$. Subtracting the second equation from the first we obtain $-3v_1=6$, so $v_1=-2$, which implies $v_2=1$. Therefore $v=\binom{-2}{1}$ is the only critical point. We now linearize the system. Define $\tilde{\mathbf{x}}=\mathbf{x}-v$. Then

$$\begin{aligned}\frac{d\tilde{\mathbf{x}}}{dt}&=\begin{pmatrix}-1 & -1\\ 2 & -1\end{pmatrix}\mathbf{x}+\begin{pmatrix}-1\\ 5\end{pmatrix}=\begin{pmatrix}-1 & -1\\ 2 & -1\end{pmatrix}(\tilde{\mathbf{x}}+v)+\begin{pmatrix}-1\\ 5\end{pmatrix}\\ &=\begin{pmatrix}-1 & -1\\ 2 & -1\end{pmatrix}\tilde{\mathbf{x}}+\underbrace{\begin{pmatrix}-1 & -1\\ 2 & -1\end{pmatrix}v+\begin{pmatrix}-1\\ 5\end{pmatrix}}_{=0}.\end{aligned}$$

Finally,

$$\det \begin{pmatrix} -1-\lambda & -1 \\ 2 & -1-\lambda \end{pmatrix} = (1+\lambda)^2 + 2.$$

Therefore $\lambda = -1 \pm i\sqrt{2}$, from which we conclude that $\binom{0}{0}$ is a spiral sink for $\tilde{\mathbf{x}}$, which is equivalent to the statement that v is a spiral sink for $\mathbf{x}$. This critical point is asymptotically stable.