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PDE FOR APPS F 08, MIDTERM I

- (1) Consider the function $f(x) = \frac{x}{1-3x}$. Find the Taylor expansion of f about $x_0 = 0$ and its radius of convergence.

Using the Taylor expansion for the geometric series we have

$$f(x) = x \sum_{n=0}^{\infty} 3^n x^n = \sum_{n=1}^{\infty} 3^{n-1} x^n.$$

By the ratio rule, the radius of convergence is $\lim_{n \rightarrow \infty} \frac{a_n}{a_{n+1}} = \frac{1}{3}$.

- (2) Solve the problem

$$x^2 y'' + 3xy' + 5y = 0, y(0) = y(1) = 0.$$

(Note: this is an Euler equation).

The indicial equation is $F(r) = r^2 + 2r + 5 = 0$. Roots are $r_{1,2} = \frac{-2 \pm \sqrt{-16}}{2} = -1 \pm 2i$. Therefore the general solution is of the form

$$y(x) = |x|^{-1} (c_1 \cos(2 \ln |x|) + c_2 \sin(2 \ln |x|)).$$

The function y has a limit (and can be defined) at 0 only if $c_1 = c_2 = 0$, in which case $y(x) = 0$ for all x , including for $x = 1$ (so the second boundary condition $y(1) = 0$ is redundant).

- (3) Let $g(t) = u_0(t)t - u_1(t)(t - 1)$.

(Recall that for any fixed c , the function $u_c(t)$ is equal to 0 for $t < c$ and is equal to 1 for $t \geq c$).

- (a) Find $G(s)$, the Laplace transform of g , explicitly.

Note that

$$g(t) = \begin{cases} 0 & t < 0 \\ t & 0 \leq t < 1 \\ 1 & t \geq 1. \end{cases}$$

Therefore

$$G(s) = \int_0^1 e^{-st} t dt + \int_1^{\infty} e^{-st} dt = -\frac{e^{-st} t}{s} \Big|_0^1 + \frac{1}{s} \int_0^1 e^{-st} dt + \frac{e^{-s}}{s} = \frac{1 - e^{-s}}{s^2}.$$

Alternatively, recall that $\mathcal{L}(t)(s) = \frac{1}{s^2}$ (by integration by parts, for example), and that $\mathcal{L}(u_c(t)f(t-c))(s) = e^{-cs} \mathcal{L}(f)(s)$. Therefore

$$G(s) = \frac{1}{s^2} - \frac{e^{-s}}{s^2} = \frac{1 - e^{-s}}{s^2}.$$

- (b) Consider the problem

$$y'' + 3y' - y = g(t), y(0) = y'(0) = 1.$$

Find Y , the Laplace transform of y . Please write " $G(s)$ " for the Laplace transform of g , rather than writing the explicit expression for G you obtained in the first part (this is to avoid repeating the same mistake twice).

We have

$$s^2 Y(s) - sy(0) - y'(0) + 3sY(s) - 3y(0) - Y(s) = G(s),$$

So $(s^2 + 3s - 1)Y(s) = (s + 4) + G(s)$. Hence

$$Y(s) = \frac{(s + 4) + G(s)}{s^2 + 3s - 1}.$$

(4)

- (a) Recall that the Laplace transform of the constant function $\mathbf{1}$ (equal to 1 for all t) is $\frac{1}{s}$. DERIVE the Laplace transform for the function t^2 from the Laplace transform of $\mathbf{1}$ using ONLY the formula for the Laplace transform for the (second) derivative of a function.

We have $\mathcal{L}(f''(t))(s) = s^2\mathcal{L}(f)(s) - sf(0) - f'(0)$. Since $\mathbf{1} = \frac{d^2}{dt^2}(\frac{1}{2}t^2)$, it follows that

$$\frac{1}{s} = \mathcal{L}(\mathbf{1})(s) = s^2\mathcal{L}(\frac{1}{2}t^2)(s) - 0 - 0 = \frac{s^2}{2}\mathcal{L}(t^2)(s).$$

Hence

$$\mathcal{L}(t^2)(s) = \frac{2}{s^3}.$$

- (b) The function $F(s) = \frac{(s+10)}{s^2+4s+40}$ is a Laplace transform of some continuous function f . Find f .

(You may use the formulas $\mathcal{L}(\sin(at))(s) = \frac{a}{s^2+a^2}$, $\mathcal{L}(\cos(at))(s) = \frac{s}{s^2+a^2}$)

Rewrite F by completing to a square

$$F(s) = \frac{s + 2 + 8}{(s + 2)^2 + 36} = \mathcal{L}(\cos(6t))(s + 2) + \frac{4}{3}\mathcal{L}(\sin(6t))(s + 2).$$

Recall that $\mathcal{L}(h(t))(s - c) = \mathcal{L}(e^{ct}h(t))(s)$, therefore

$$F(s) = \mathcal{L}(e^{-2t} \cos(6t))(s) + \frac{4}{3}\mathcal{L}(e^{-2t} \sin(6t))(s),$$

and then

$$f(s) = e^{-2t} \left(\cos(6t) + \frac{4}{3} \sin(6t) \right).$$