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PDE for Apps F 08, Midterm II

1. Let f be a bounded continuous function on $[0, \infty)$. Express the solution of the following equation as a convolution integral

$$4y'' + 4y' + 17y = f(t), \ y(0) = y'(0) = 0.$$

(It may help you to complete $4s^2 + 4s + 17$ to a square)

2. Suppose that the function $g(t) : [0, \infty) \to (0, \infty)$ is given by $g(t) = \mathbf{1} + \mathbf{1} * \mathbf{1} + \mathbf{1} * \mathbf{1} * \mathbf{1} + \ldots$. Calculate the Laplace transform of g explicitly, and use that to find g.
($\mathbf{1}$ is the constant function $\mathbf{1}(t) = 1$ for all t)

3. (a) Find the Fourier series of the function$f(x) = |x|$ on the interval $[-1, 1]$.
   (b) By letting $x = 0$ in the series you obtained in the first part, compute $\sum\limits_{n \text{ odd } \geq 1} \frac{1}{n^2}$.

4. Consider the following boundary value problem:

$$(*) \begin{cases} 100u_{xx} = u_t & 0 < x < 1, t > 0 \ (\text{differential equation}) \\ u(0,t) = 1, \ u(1,t) = 2 & t \geq 0 \ (\text{boundary conditions}) \\ u(x,0) = 1 + x + \sin(\pi x) & 0 \leq x \leq 1 \ (\text{initial condition}). \end{cases}$$

Solve the boundary value problem $(*)$ using the steps below:

   (a) Write the equation and the boundary conditions the steady state solution v should satisfy, and find the steady state explicitly.
   (b) Let $w(x,t) = u(x,t) - v(x)$. Find what boundary value problem w solves (the boundary value problem consists of: a differential equation, boundary conditions and initial condition).
   (c) Suppose that $w(x,t) = \sum_{n=1}^{\infty} c_n w_n(x,t)$, where c_n are constants and the functions are given by $w_n(x,t) = X(x)T(t)$, where X and T depend on n. Find the equations as well as the boundary conditions (if any) that X and T satisfy.
   (d) Solve the equations you obtained in the previous step.
   (e) Use the initial condition on w to obtain the constants c_n.
   (f) Write the formula you obtained for $u(x,t)$.

Solutions

1. Write $4s^2Y(s) + 4sY(s) + 17Y(s) = F(s)$. Therefore $Y(s) = \frac{F(s)}{4s^2+4s+17}$. Now

$$4s^2 + 4s + 17 = 4((s + \frac{1}{2})^2 - \frac{1}{4} + \frac{17}{4}) = 4((s + \frac{1}{2})^2 + 4).$$

Hence

$$\frac{1}{4s^2 + 4s + 17} = \frac{1}{8}\mathcal{L}(\sin(2t))(s + \frac{1}{2}) = \mathcal{L}(\frac{e^{-t/2}}{8}\sin(2t))(s) = \mathcal{L}(g(t))(s) = G(s).$$

In particular, $Y = F * G = \mathcal{L}((g * f)(t))$, therefore

$$y(t) = (g * f)(t) = \frac{1}{8}\int_0^t e^{-(t-s)/2}\sin(2(t-s))f(s)ds.$$

2.

$$G(s) = \frac{1}{s} + \frac{1}{s^2} + \cdots = \frac{1}{s}\left(\frac{1}{1 - s^{-1}}\right) = \frac{1}{s-1} = \mathcal{L}(\mathbf{1})(s-1) = \mathcal{L}(e^t\mathbf{1})(s) = \mathcal{L}(e^t)(s).$$

Therefore $g(t) = e^t$.

3. For $n \geq 0$, $a_n = 2\int_0^1 x\cos(n\pi x)dx$. Then $a_0 = 1$ and for $n \geq 1$,

$$a_n = 2\frac{x\sin(n\pi x)}{n\pi}\bigg|_0^1 - \frac{2}{n\pi}\int_0^1 \sin(n\pi x)dx = 0 - \frac{2}{(n\pi)^2}(-\cos(n\pi x))\bigg|_0^1 = \begin{cases} -\frac{4}{(n\pi)^2} & n \text{ odd} \\ 0 & n \text{ even.} \end{cases}$$

Therefore

$$0 = f(0) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n = \frac{1}{2} - \frac{4}{\pi^2}\sum_{n \text{ odd}} \frac{1}{n^2}$$

In particular

$$\sum_{n \text{ odd}} \frac{1}{n^2} = \frac{\pi^2}{8}.$$

4. (a)

$$\begin{cases} 100v_{xx} = 0 & 0 < x < 1 \\ v(0) = 1,\ v(1) = 2. \end{cases}$$

Hence $v(x) = 1 + x$.

(b)

$$\begin{cases} 100w_{xx} = w_t & 0 < x < 1, t > 0 \ (\text{differential equation}) \\ w(0,t) = 0,\ w(1,t) = 0 & t \geq 0 \ (\text{boundary conditions}) \\ w(x,0) = \sin(\pi x) & 0 \leq x \leq 1 \ (\text{initial condition}). \end{cases}$$

(c)

$$\begin{cases} X''(x) + \lambda X(x) = 0 \qquad X(0) = X(1) = 0 \\ T'(t) + 100\lambda T(t) = 0. \end{cases}$$

(d) $\lambda = -(n\pi)^2,\ X(x) = \sin(n\pi x),\ n = 1, 2, \ldots$ and $T(t) = e^{-100n^2\pi^2 t}$.

(e) $\sin(\pi x) = w(x, 0) = \sum_{n=1}^{\infty} c_n \sin(n\pi x)$. Therefore $c_1 = 1$ and $c_n = 0$ for all $n \geq 2$. In particular

$$w(x, t) = e^{-100\pi^2 t} \sin(\pi x) \Rightarrow u(x, t) = e^{-100\pi^2 t} \sin(\pi x) + 1 + x.$$