Measure and Integration (Math 5111) S09
Final Exam

- Answer exactly 5 out of the following 7 problems. **You must** answer Problem 6 or Problem 7. Mark on the form the problems you want to be graded. Otherwise, first 5 solutions will be graded.
- Results proved in lectures should be applied without proof. In this case you have to explain precisely and unambiguously what result you are using.

Below X is a nonempty set and $(X, \mathcal{F}, \mu)$ denotes a measure space.

1. (σ-algebras and measures) Let $\{E_j\}_{j=1}^{\infty}$ be a sequence of sets in $\mathcal{F}$. Define $\limsup_{j\to\infty} E_j = \cap_{n=1}^{\infty} \cup_{j\geq n} E_j$.
   (a) Suppose that $\mu(\cup_{j=1}^{\infty} E_j) < \infty$. Prove that $\mu(\limsup_{j\to\infty} E_j) \geq \limsup_{j\to\infty} \mu(E_j)$.
   (b) Prove that if $\sum_{j=1}^{\infty} \mu(E_j) < \infty$ then $\mu(\limsup E_j) = 0$.
2. (Outer measures) Suppose that ν_0 is a finite premeasure ($\nu_0(X) < \infty$) on an algebra $\mathcal{A}_0$ of sets in X. Let ν^* denote the induced outer measure.
   (a) Let $\mathcal{A}_\sigma$ denote the set of collection of countable unions of sets in $\mathcal{A}_0$. Prove that for any $E \subset X$,
   $$\nu^*(E) = \inf\{\nu^*(A) : E \subset A,\ A \in \mathcal{A}_\sigma\}.$$
   (b) Let $\mathcal{A}_{\sigma\delta}$ denote the collection of countable intersections of sets in $\mathcal{A}_\sigma$. Prove that for any $E \subset X$ there exists $A \in \mathcal{A}_{\sigma\delta}$ such that $E \subset A$ and $\nu^*(E) = \nu^*(A)$.
   (c) Let $\mathcal{A}^*$ denote the σ-algebra of ν^*-measurable sets. Prove
   $$\mathcal{A}^* = \{E \subset X : \exists A \in \mathcal{A}_{\sigma\delta},\ E \subset A, \ \text{ and } \nu^*(A \backslash E) = 0\}.$$
3. (Measurable functions) Suppose that $f_n : X \to \overline{\mathbb{R}},\ n = 1, 2, \ldots,$ are measurable functions.
   (a) Prove that $\liminf_{n\to\infty} f_n$ is measurable.
   (b) Prove that $\{x : \lim_{n\to\infty} f_n(x)$ exists and is finite$\}$ is measurable.
   (c) Let $(X, \overline{\mathcal{F}}, \overline{\mu})$ be the completion of $(X, \mathcal{F}, \mu)$. Prove that if $g : X \to \mathbb{R}$ is $\overline{\mathcal{F}}$-measurable, then there exists $\mathcal{F}$-measurable f such that $f = g$, μ a.e. . **I stress**: "μ a.e." opposed to "$\overline{\mu}$ a.e."
4. (Integration of nonnegative functions)
   (a) State Fatou's Lemma.
   (b) State the Monotonce Convergence Theorem.
   (c) Prove that Fatou's Lemma and the Monotone Convergence Theorem are equivalent (that is either one implies the other).
   (d) Give an explicit example of a sequence of functions for which the conclusion of Fatou's Lemma does not hold.
5. (Applications of integration) Let $(\mathbb{R}, \mathcal{L}, m)$ denote the real line equipped with Lebesgue σ-algebra and the Lebesgue measure. For $f \in L^1(m)$ define $\hat{f}(t) = \int f(x)e^{itx}dm(x)$. Prove the following
   (a) $\hat{f}$ is continuous.
   (b) $\lim_{|t|\to\infty} \hat{f}(t) = 0$.
   (c) If f vanishes outside a bounded interval, then $\hat{f}$ is differentiable infinitely many times and and $\hat{f}^{(k)}(t) = i^k \int x^k f(x)e^{itx}dm(x)$.
6. (Modes of convergence)
   (a) Define the term: Cauchy sequence in measure.
   (b) State the Dominated Convergence Theorem for convergence in measure.
   (c) Prove the above theorem.

7. (Radon-Nikodym and Absolute Continuity)
   (a) Let $\mathcal{G}$ be a σ-algebra contained in $\mathcal{F}$. Prove that for any $f \in L^1$ there exists a unique $\mathcal{G}$-measurable function $f_{\mathcal{G}}$ such that for all $A \in \mathcal{G}$, $\int_A f d\mu = \int_A f_{\mathcal{G}} d\mu$.
   (b) Suppose that $f \in L^1_{loc}(m)$ satisfies $\int_{(a,b)} f(x)dm(x) = 0$ for any pair of points $a = k2^{-n}, b = j2^{-n}$ where $n \in \mathbb{N}, k, j \in \mathbb{Z}$. Prove that $f = 0$ m a.e. .