ENERGY AND LAPLACIAN ON HANOI-TYPE FRACTAL QUANTUM GRAPHS

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Abstract. We study energy and spectral analysis on compact metric spaces which we refer to as fractal quantum graphs. These are spaces that can be represented as a (possibly infinite) union of 1-dimensional intervals and a totally disconnected (possibly uncountable) compact set, which roughly speaking represents the set of junction points. These spaces include classical quantum graphs and fractal spaces such as the Hanoi attractor, which is weakly self-similar. We begin with proving the existence of a resistance form on the Hanoi attractor, and discuss the spectral asymptotics of the Laplacians corresponding to weakly self-similar measures. We then state and prove the existence of resistance forms on general fractal quantum graphs. Finally, we prove spectral asymptotics for a large class of weakly self-similar fractal quantum graphs.

1. Introduction

This paper proves the existence of resistance forms on fractal quantum graphs, and in the special case where there is some form of weakly self-similar measure, we prove asymptotics of the eigenvalue counting function for the Laplacian induced by this measure.

Resistance forms are Dirichlet forms on which effective resistance between points defines a metric. Resistance forms have been very useful in the study of analysis on fractals from an intrinsic point of view, starting with Jun Kigami’s work on post-critically finite self-similar fractals in [Kig01, Chapter 2].

In [Kig12], resistance forms are developed as the limit of finite approximating electrical networks. The main challenge in proving the existence of a resistance form is proving that the limiting topology on these resistance networks agrees with the original space. However, most examples of resistance forms come from self-similar cases, [HMT06, FST06, BCF+07].

The text [Tep08] discusses the generalization of resistance forms to spaces which aren’t self-similar. Following along these lines, we prove the existence of a resistance form on general spaces which have no a priori self-similarity. In [HT12, IRT12] a general theory of geometric analysis is developed for Dirichlet spaces in general, in [HKT13] this is applied to resistance forms to talk about length structures and differential equations on these spaces.

We are interested in spectral asymptotics specifically to understand the spectral dimension of these fractals. This dimension determines properties of the Laplacian and the diffusion process generated by this Laplacian [bAH00]. This quantity also determines physical aspects of the space [ADT09].

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In proving spectral asymptotics, we assume some self-similarity of the space. This is for two reasons. First, we are faced with the choice of the measure. The Hanoi attractor for parameter $\alpha \in (0,1/3)$ has Hausdorff dimension strictly greater than 1, which complicates the analysis. However, all but a completely disconnected/topologically 0 dimensional set is locally isometrically isomorphic to an interval. To deal with this issue we introduce new weakly-self similar measures.

The second reason we need to assume self-similarity, is that we require techniques from [KL93, Kaj10]. These arguments, informally speaking, use the fact that small-scale metric properties correspond to larger eigenvalues. From this, self-similarity is critical in achieving spectral asymptotics, as it allows us to infer properties of the fractal at arbitrarily small scales.

The use of quantum graphs in chaotic (fractal) frameworks has been been discussed in [KS02, KS03, Kuc04]. Fractal networks in particular have been of interest in the study of superconductivity [Ale83, AH83]. The work here can be applied to the dendrite fractals considered in [Kig95]. Note also recent topological results on very similar spaces in [Geo] as well as construction of Brownian motion on them in [GK].

We start analyzing this problem for the Hanoi attractor of parameter $\alpha$ that we denote by $K_\alpha$. These are non self-similar fractals, where the parameter $\alpha$ can be understood as a “critical metric parameter”: On the one hand, $\alpha$ is the length of the three longest segments (see Figure 1). On the other hand, we may call it critical since in the case $\alpha = 0$, $K_\alpha$ coincides with the Sierpiński gasket, if $\alpha \in (0, 1/3)$, then $K_\alpha$ has fractional Hausdorff dimension $\frac{\ln 3}{\ln 2 - \ln (1-\alpha)}$ and the case $\alpha \in (1/3, 1)$ reveals a 1-dimensional object. We refer to [ARF12] for geometric results about these attractors, which can also be understood as graph-directed fractals, introduced in [MW88] and treated analytically in [HN03].

![Figure 1. The Hanoi attractor $K_\alpha$.](image)

This paper is organized as follows: After recalling some basics on quantum graphs in Section 2, we fix $\alpha \in (0, \frac{1}{3})$ and set $X := K_\alpha$. Section 3 is devoted to the definition of a sequence of metric graphs $(X_n)_{n \in \mathbb{N}_0}$ that approximates $X$. In Section 4, we establish the energy on $X$ that comes from the initial expression

$$\mathcal{E}(u, v) := \int_X u' v' \, dx \quad u, v \in \bigcup_{n \in \mathbb{N}_0} H^1(X_n),$$

where $H^1(X_n)$ denotes the Sobolev space on the metric graph $X_n$. Functions belonging to this space are considered to be the continuous representative whenever possible. We prove that for a suitable domain $\text{Dom} \mathcal{E}$ we have

**Theorem 1.1.** $(\mathcal{E}, \text{Dom} \mathcal{E})$ is a resistance form.
Section 5 deals with the spectral asymptotics of the Laplacian associated to the Dirichlet form induced by the resistance form $(\mathcal{E}, \text{Dom} \mathcal{E})$. For this we introduce a locally finite regular measure $\mu$ on $X$, where a "critical measure parameter" $\beta$ appears. This parameter gives the measure of each of the three lines of length $\alpha$ in $X$. For $\mu$ to be a probability measure, it has to satisfy

\[ 1 = \mu(X) = 3\beta + 3\mu(X'), \]

where $X'$ denotes a first-level copy of $X$ (see Figure 2). If we set $\mu(X') =: s$, then we get from (1.1) that

\[ s = \frac{1 - 3\beta}{3} \]

and hence $0 < \beta < \frac{1}{3}$. Note that if $\beta = \frac{1}{3}$, then $\mu(X') = 0$, which is undesirable as the support of $\mu$ would not be all of $X$. These assumptions will be briefly recalled at the beginning of Section 5.

The most important result in this section is the calculation of the spectral dimension of $X$. In order to obtain this we need the following proposition, where we introduce together with the parameter $s$ defined in (1.2) the quantity $r = \frac{1 - \alpha}{2}$ which is the scaling length of the three triangles shadowed in Figure 2.

**Proposition 1.2.** Let $rs = \frac{1}{9}(1 - \alpha)(1 - 3\beta)$. There exist constants $C_1, C_2 > 0$ and $x_0 > 0$ such that

(i) if $0 < rs < \frac{1}{9}$, then

\[ C_1 x^{\frac{1}{2}} \leq N_D(x) \leq N_N(x) \leq C_2 x^{\frac{1}{2}}, \]

(ii) if $rs = \frac{1}{9}$, then

\[ C_1 x^{\frac{1}{2}} \log x \leq N_D(x) \leq N_N(x) \leq C_2 x^{\frac{1}{2}} \log x, \]

(iii) if $\frac{1}{9} < rs < \frac{1}{6}$, then

\[ C_1 x^{-\frac{\log 3}{\log rs}} \leq N_D(x) \leq N_N(x) \leq C_2 x^{-\frac{\log 3}{\log rs}} \]

for all $x > x_0$.

Here $N_D(x)$ are $N_N(x)$ denote the eigenvalue counting function of the Laplacian associated to the Dirichlet form induced by $(\mathcal{E}, \text{Dom} \mathcal{E})$ under Dirichlet –resp. Neumann– boundary conditions.

Notice that $rs < \frac{1}{9}$ follows directly from the fact that $\alpha, \beta \in (0, \frac{1}{3})$. From this proposition we get immediately the spectral dimension of $X$, $d_S X$. 

![Figure 2. Each first-level copy $X'$ of $X$ is contained in each shadowed part.](image)
Theorem 1.3.

\[ d_S X = \begin{cases} 
\frac{1}{\log 9}, & 0 < rs \leq \frac{1}{9}, \\
\frac{1}{9} - \log(rs), & \frac{1}{9} < rs < \frac{1}{6}.
\end{cases} \]

This result shows us that both the metric and the measure parameter strongly affect the spectral properties of the operator.

Section 6 solves the question about existence of resistance forms in a more general framework, what we will call fractal quantum graphs. These consist of a separable compact connected locally connected space \((X,d)\) together with a sequence of lengths \(\{\ell_k\}_{k=1}^{\infty} \subset (0,\infty)\) and isometries \(\Phi_k: [0,\ell_k] \to X\) such that \(X \setminus \bigcup_{k=1}^{\infty} \Phi_k((0,\ell_k))\) is totally disconnected.

Under mild conditions on the isometries \(\Phi_k\), we construct a resistance form \((\mathcal{E}, \text{Dom}\, \mathcal{E})\) on \(\Omega\), which is the completion of \(D_*\) with respect to the effective resistance metric associated to \(\mathcal{E}\), and \(D_*\) is a dense subset of \(X\) with respect to the Euclidean metric.

Since \(\Omega\) cannot be in general identified with \(X\), the goal of the section is proving that

**Proposition 1.4.** \(\Omega\) is homeomorphic to \(X\).

To this purpose, we will add some more structure and assumptions on \(X\). Note that both quantum graphs and Hanoi attractors (as well as generalized Hanoi-type quantum graphs treated later) also satisfy these assumptions.

Finally, the last section presents the so-called generalized Hanoi-type quantum graphs with parameters \(N_0\) and \(\alpha\) that we denote by \(X_{N_0,\alpha}\). In this case, \(N_0\) can be understood as a “dimension parameter” since \(\dim_H X_{N_0,\alpha} \leq N_0 - 1\). The parameter \(\alpha\) is again the length of the longest segments in \(X_{N_0,\alpha}\) and it will be chosen to lie in the interval \((0, \frac{N_0 - 2}{N_0})\) in order to get \(N_0 - 2 < \dim_H X_{N_0,\alpha} < N_0 - 1\) and thus a fractal object.

The construction of the resistance form \((\mathcal{E}, \text{Dom}\, \mathcal{E})\) in this case is carried out in the same way as in Section 4. In order to get a Dirichlet form out of it, we introduce a measure on \(X_{N_0,\alpha}\) that depends again on a “measure parameter” \(\beta\) which measures the segments of length \(\alpha\).

Following analogous arguments as in Section 5, we study the spectral asymptotics of the Laplacian associated to the Dirichlet form induced by \((\mathcal{E}, \text{Dom}\, \mathcal{E})\) and get

**Proposition 1.5.** Let \(r = \frac{1-\alpha}{2}\) and \(s = \frac{2-N_0(N_0-1)\beta}{2N_0}\). There exist constants \(C_1, C_2 > 0\) and \(x_0 > 0\) such that

(i) if \(0 < rs < \frac{1}{N_0}\), then

\[ C_1 x^{\frac{\alpha}{2}} \leq N_D(x) \leq N_N(x) \leq C_2 x^{\frac{\alpha}{2}}, \]

(ii) if \(rs = \frac{1}{N_0}\), then

\[ C_1 x^{\frac{\alpha}{2}} \log x \leq N_D(x) \leq N_N(x) \leq C_2 x^{\frac{\alpha}{2}} \log x, \]
(iii) if \( \frac{1}{N_0^2} < rs < \frac{1}{2N_0} \), then
\[
C_1 x^{\log N_0} - \log(rs) \leq N_D(x) \leq N_N(x) \leq C_2 x^{\log N_0} \]
for all \( x > x_0 \).

This leads to the following expression of the spectral dimension of \( X_{N_0} \).

**Theorem 1.6.** It holds that
\[
d_{S} X = \begin{cases} 
1 & 0 < rs \leq \frac{1}{N_0^2}, \\
\frac{\log N_0^2}{-\log(rs)} & \frac{1}{N_0^2} < rs < \frac{1}{2N_0}, \\
\end{cases}
\]

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## 2. Quantum graph basics

A graph \( G = (V, E, \partial) \) is a discrete set of vertices \( V \) along with a set of edges \( E \), with a map \( \partial : E \to V \times V \), defined by \( \partial e := (\partial^+ e, \partial^- e) \). For \( v \in V \), define \( E^\pm_v := \partial^\pm_1 \{v\} \) and \( E_v := E^-_v \cup E^+_v \), that is to say all the edges that begin or end at \( v \).

A weighted graph has the additional structure of \( r : E \to (0, \infty) \). The weight, or conductance, of an edge \( e \) is the quantity \( \frac{1}{r(e)} \), thus \( r(e) \) is the resistance of the edge \( e \).

A metric graph \( G_{\text{met}} \), is the CW 1-complex with set of 0-cells \( V \) and the set of 1-cells indexed by the edges with endpoints given by \( \partial^\pm \). This set is given a metric by considering \( \Phi_e : I_e \to G_{\text{met}} \) where \( I_e = [0, r(e)] \) and its image is the 1-cell associated to edge \( e \), where \( \Phi_e|_{(0, r(e))} \) is a homeomorphism.

We also give \( G_{\text{met}} \) a measure \( m \) which is induced by \( \Phi_e \). We shall define the space of \( L^2 \) and Sobolev functions on \( G_{\text{met}} \) by
\[
L^2(G_{\text{met}}) = \bigoplus_{e \in E} L^2(I_e) \quad \text{and} \quad H^n(G_{\text{met}}) = \bigoplus_{e \in E} H^n(I_e),
\]
where \( L^2(I_e) \) and \( H^n(I_e) \) are classical \( L^2 \) and Sobolev spaces on the interval \( I_e \). We identify the above with functions on \( G_{\text{met}} \) by the maps \( \Phi_e \) (notice that \( V \) is a set of measure 0).

## 3. Definitions of Hanoi attractors

In this section we briefly recall the definition of Hanoi attractors and approximate them by quantum graphs.

Let us fix any \( \alpha \in (0, \frac{1}{3}) \) and consider in \( \mathbb{R}^2 \) the points
\[
p_1 := (0, 0), \quad p_2 := \left( \frac{1}{2}, \frac{\sqrt{3}}{2} \right), \quad p_3 := (1, 0),
p_4 := \frac{p_2 + p_3}{2}, \quad p_5 := \frac{p_1 + p_3}{2}, \quad p_6 := \frac{p_1 + p_2}{2},
\]
which are the fixed points of the mappings
\[ G_{\alpha,i} : \mathbb{R}^2 \to \mathbb{R}^2 \]
\[ x \mapsto A_i(x - p_i) + p_i \]
i = 1, \ldots, 6,\]
where
\[ A_1 = A_2 = A_3 = \frac{1 - \alpha}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad A_4 = \frac{\alpha}{4} \begin{pmatrix} 1 & -\sqrt{3} \\ -\sqrt{3} & 3 \end{pmatrix}, \]
\[ A_5 = \alpha \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad A_6 = \frac{\alpha}{4} \begin{pmatrix} 1 & \sqrt{3} \\ \sqrt{3} & 3 \end{pmatrix}. \]
The parameter \( \alpha \) is chosen to lie in the interval \((0, \frac{1}{3})\) in order to avoid overlaps and assure getting a fractal object. Since the contraction ratios \( r_i \) of each \( G_{\alpha,i} \) satisfy
\[ 0 < r_i < 1, \{G_{\alpha,i}\}_{i=1}^6 \]
is a family of contractions and the iterated function system \( \{\mathbb{R}^2; G_{\alpha,1}, \ldots, G_{\alpha,6}\} \)
leads to a unique non-empty compact set \( K_\alpha \subset \mathbb{R}^2 \)
such that
\[ K_\alpha = \bigcup_{i=1}^6 G_{\alpha,i}(K_\alpha). \]
This set is called the \textit{Hanoi attractor of parameter} \( \alpha \). Because in the following the parameter \( \alpha \) is arbitrary but fixed, to simplify notation we will write \( X := K_\alpha \) and \( F_i := G_{\alpha,i} \) for each \( i = 1, \ldots, 6 \).

Note that \( X \) is not strictly self-similar since the contractions \( F_4, F_5 \) and \( F_6 \) are not similitudes. However, these fractals still have some weakly self-similarity due to the similitudes \( F_1, F_2 \) and \( F_3 \). In order to approximate \( X \) by metric graphs, we will only use the similitudes and their corresponding fix points \( p_1, p_2, p_3 \) and forget about \( p_4, p_5 \) and \( p_6 \).

Now we introduce some useful notations and definitions. We denote by \( \mathcal{A} \) the alphabet on the symbols 1, 2, 3. For each word \( w = w_1 \cdots w_n \in \mathcal{A}^n, n \in \mathbb{N} \), we define
\[ F_w(x) := F_{w_1} \circ F_{w_2} \circ \cdots \circ F_{w_n}(x), \quad x \in \mathbb{R}^2 \]
and \( F_{w_0} := \text{id}_{\mathbb{R}^2} \) for the empty word \( w_0 \).

\textbf{Definition 3.1.} For any \( n \in \mathbb{N}_0 \), we define the vertex set
\[ V_n := \bigcup_{w \in \mathcal{A}^n} F_w(\{p_1, p_2, p_3\}) \]
and the edge set \( E_n := T_n \cup J_n \), where
\[ T_n := \{\{x, y\} \mid \exists w \in \mathcal{A}^n \text{ s.t. } x, y \in F_w(V_0)\}, \]
\[ J_n := \{\{x, y\} \mid \exists 0 < k < n, w \in \mathcal{A}^{k-1} \text{ s.t. } x = F_{w_j}(p_i), y = F_{w_j}(p_j), i, j \in \mathcal{A}, i \neq j\}. \]
Moreover, let \( r : E_n \to (0, \infty) \) be the weight function given by the edge length, i.e.
\[ r(e) := \begin{cases} 
(\frac{1-\alpha}{2})^n, & \text{for } e \in T_n, \\
\alpha \left(\frac{1-\alpha}{2}\right)^k, & \text{for } e \in J_n, e = \{F_{w_j}(p_i), F_{w_j}(p_j)\}, w \in \mathcal{A}^{k-1}.
\end{cases} \]
Then \( X_n := (V_n, E_n, \partial) \) together with \( r \) defines a metric graph. We may take any orientation \( \partial \).
Thus we have two different types of edges in \( E_n \): on one hand, \( T_n \) contains “triangle-type” edges, i.e. edges that build a triangle. On the other hand, \( J_n \) denotes the set of “joining-type” edges, since they join the triangles built by the edges in \( T_n \).

We equip these graphs with the measure \( m \) introduced in Section 2. Notice that this measure coincides with the 1–dimensional Lebesgue measure. Thus \( (X_n)_{n \in \mathbb{N}_0} \) is a sequence of metric graphs that approximates \( X \) as Figure 3 suggests and we may write

\[
X = \text{cl} \left( \bigcup_{n \in \mathbb{N}_0} X_n \right),
\]

where \( \text{cl}(\cdot) \) means closure with respect to the Euclidean metric.

**Remark 1.**

(i) The space \((X, m)\) is a \( \sigma \)-finite measure space that is not finite because due to the choice of \( \alpha \) we have that

\[
m(X) = \sum_{n=0}^{\infty} \sum_{e \in E_n} m(I_e) \geq \sum_{n=0}^{\infty} \sum_{e \in J_n} m(I_e) = 3\alpha \sum_{n=1}^{\infty} 3^n \left( \frac{1 - \alpha}{2} \right)^n = +\infty.
\]

Recall that \( I_e = [0, r(e)] \) is the interval associated with the edge \( e \).

(ii) If we define the sets

\[
D_n := \left\{ \Phi_e \left( \frac{r(e)k}{2^n} \right) \mid e \in J_n, \, 0 \leq k \leq 2^n \right\},
\]

then \( V_n := \bigcup_{n \in \mathbb{N}_0} D_n \) is dense in \( X \) with respect to the Euclidean metric. Also note that \( V_n \subseteq D_n \forall n \in \mathbb{N} \).

In order to get a quantum graph out of the metric graph \( X_n \), we need to introduce a differential operator, which is obtained through the energy \( \mathcal{E} \), that we define in the following. The crucial point here is the choice of the domains \( \mathcal{F}_n \), whose functions are everywhere constant except in finitely many “joining-type” edges. For each \( n \geq 0 \), the set of joining-type edges in level \( n \) is defined by \( J_n := \bigcup_{e \in J_n} \Phi_e(I_e) \).

**Definition 3.2.** Consider the non-negative symmetric bilinear form given by

\[
\mathcal{E}(u, v) := \int_X u' v' \, dx \quad u, v \in \bigcup_{n \in \mathbb{N}_0} H^1(X_n)
\]

and define

\[
\mathcal{F}_n := \{ u : X \to \mathbb{R} \mid \forall n \in \mathbb{N}_0, \, u|_{x_n} \in H^1(X_n) \text{ and } u|_e \equiv c_e \forall e \in X \setminus J_n \},
\]

where \( c_e \) is some constant that only depends on \( e \). The bilinear form \( \mathcal{E} \) together with the domain \( \mathcal{F} \) is called the energy of the \( n \)-th approximation of \( X \).
Remark 2. Note that \( \mathcal{E}(u, u) < \infty \) for all \( u \in \mathcal{F}_n \) and \( n \in \mathbb{N}_0 \).

4. Energy on Hanoi attractor is a resistance form

We refer to [Kig12] for the definition and basic properties of resistance forms, some of them we recall here.

Definition 4.1. Let \( X \) be a set. A pair \((\mathcal{E}, \text{Dom } \mathcal{E})\) is called a resistance form if

1. \( \mathcal{E} \) is a non-negative symmetric bilinear form, \( \text{Dom } \mathcal{E} \) is a linear subspace of \( \ell(X) := \{ u: X \to \mathbb{R} \} \) that contains constants and \( \mathcal{E}(u, u) = 0 \) if and only if \( u \) is constant on \( X \).
2. If \( \sim \) is the equivalence relation in \( \text{Dom } \mathcal{E} \) where \( u \sim v \) iff \( u - v \) is constant, then \((\text{Dom } \mathcal{E}/\sim, \mathcal{E})\) is a Hilbert space.
3. For any two points \( x \neq y \) in \( X \), there exists \( u \in \text{Dom } \mathcal{E} \) such that \( u(x) \neq u(y) \).
4. For any \( p, q \in X \)

\[
\sup \left\{ \frac{(u(p) - u(q))^2}{\mathcal{E}(u, u)} \mid u \in \text{Dom } \mathcal{E}, \mathcal{E}(u, u) > 0 \right\} < \infty
\]

We shall denote this supremum by \( R(p, q) \) and call it the effective resistance between \( p \) and \( q \).

5. (Markov property) For any \( u \in \text{Dom } \mathcal{E}, \tilde{u} \in \text{Dom } \mathcal{E} \), where

\[
\tilde{u}(p) := \begin{cases} 
0 & \text{if } u(p) \leq 0, \\
u(p) & \text{if } 0 < u(p) < 1, \\
1 & \text{if } u(p) \geq 1.
\end{cases}
\]

Additionally, it holds that \( \mathcal{E}(\tilde{u}, \tilde{u}) \leq \mathcal{E}(u, u) \)

Definition 4.2. If \((\mathcal{E}, \text{Dom } \mathcal{E})\) is a resistance form on \( X \) and \( S \) is a finite subset of \( X \), then we define the resistance form \( \text{Tr}_S \mathcal{E}: \ell(S) \times \ell(S) \to \mathbb{R} \) by

\[
\text{Tr}_S \mathcal{E}(u, v) := \inf \left\{ \mathcal{E}(v, v) \mid v|_S = u \right\}.
\]

For any \( u, v \in \ell(S) \), \( \text{Tr}_S(u, v) \) is defined applying the polarization identity.

First some metric observations:

Lemma 4.1. For points \( p, q \in X_n \), there is an integer constant \( c(p, q) > 0 \) depending only on \( p, q \) such that for any function \( u \in \mathcal{F}_n \),

\[
|u(p) - u(q)|^2 \leq \frac{\mathcal{E}(u)d_n(p, q)}{c(p, q)},
\]

where \( d_n \) is the geodesic/intrinsic distance on \( X_n \). In particular, if \( u \) is non-constant, then \( \mathcal{E}(u) := \mathcal{E}(u, u) > 0 \) and thus vanishes only on constants.

Proof. If \( p, q \) are both on the same edge, then

\[
|u(p) - u(q)|^2 = \int_p^q |u'(x)|^2 \, dx \leq \int_p^q |u'(x)|^2 \, dx \left| p - q \right| \leq \mathcal{E}(u)\left| p - q \right|.
\]

If \( p \) and \( q \) are not on the same edge, then there is \( x_0, \ldots, x_m \in X_n \) such that \( p = x_0, q = x_m \) and \( x_i \) and \( x_{i+1} \) belong to the same edge (these are the vertices which a path from \( p \) to \( q \) would pass through).
Thus
\[ |u(p) - u(q)|^2 \leq |u(x_0) - u(x_1)| + |u(x_1) - u(x_2)| + \cdots + |u(x_{m-1}) - u(x_m)| \]
\[ \leq \mathcal{E}(u) \left( \sum_{i=0}^{m-1} |x_i - x_{i+1}|^{1/2} \right)^2 \]
\[ \leq \frac{\mathcal{E}(u)}{m} \left( \sum_{i=0}^{m-1} |x_i - x_{i+1}| \right), \]
where the last inequality is by Jensen’s inequality. If we assume that \( x_i \) are the vertices transversed by the length minimizing path from \( p, q \), then we get the inequality in the lemma. Note that \( c(p, q) \) is the combinatorial length of the length minimizing path from \( p \) to \( q \).

We define the forms \( \mathcal{E}_n : \ell(D_n) \times \ell(D_n) \to \mathbb{R} \) by
\[ \mathcal{E}_n(u, u) := \inf \{ \mathcal{E}(v, v) \mid v \in \mathcal{F}_k, v|_{V_n} = u, k = 1, 2, \ldots \} \]
and for each \( u, v \in \ell(D_n) \), \( \mathcal{E}_n(u, v) \) is given by the polarization identity. Notice that this is well defined, because for any \( u \in \ell(D_n) \) there is a function \( v \in \mathcal{F}_{n+1} \) such that \( v|_{D_n} = u \).

**Proposition 4.2.** For all \( n \in \mathbb{N}_0 \), \( (\mathcal{E}_n, \ell(D_n)) \) is a resistance form and \( \text{Tr}_{D_n} \mathcal{E}_{n+1} = \mathcal{E}_n \).

We give two proofs, the first is existential.

**Proof.** (RF 1) is clear. (RF 2) follows because \( \mathcal{E}_n \) is a non-negative definite, symmetric, bilinear form on \( \ell(D_n) \) and by Lemma 4.1 \( \mathcal{E}_n(u) := \mathcal{E}_n(u, u) = 0 \) if and only if \( u \) is constant, so \( \mathcal{E}_n \) is a Hilbert space on the finite dimensional \( \ell(D_n) \).

(RF 3) Given \( x, y \in D_n \), then either \( x \) or \( y \in \Phi_e(I_e) \) for some \( e \in J_n \), or there exist \( w_1 \neq w_2 \in A^n \) such that \( x \in F_{w_1}(X) \) and \( y \in F_{w_2}(X) \). In either case there is a function in \( \mathcal{F}_n \) which separates these points.

(RF 4) This follows directly from Lemma 4.1.

(RF 5)
\[ \mathcal{E}_n(\tilde{u}) = \inf \{ \mathcal{E}(v) \mid v|_{V_n} = \tilde{u} \} \]
\[ = \inf \{ \mathcal{E}(\tilde{v}) \mid \tilde{v}|_{V_n} = \tilde{u} \} \]
\[ = \inf \{ \mathcal{E}(\tilde{v}) \mid v|_{V_n} = u \} \]
\[ \leq \inf \{ \mathcal{E}(v) \mid v|_{V_n} = u \} = \mathcal{E}_n(u) \]

An alternative proof is given along the style of that used in [BCF+07]. We start off with a network representing the first level approximation to the Hanoi attractor, that is, three triangular connections with resistance \( R \) each, connected one to the other two by a wire of resistance \( \alpha \), as indicated in Figure 4.

We assume that in this network the effective resistance between the corners of the large triangle must be the same as that in a triangular network where all connections have resistance 1. Then we get that
\[ \frac{5R}{9} + \frac{\alpha}{3} = \frac{1}{3}, \text{ thus } R = \frac{3}{5}(1 - \alpha). \]
Consider the bilinear form $\mathcal{E}$, with

$$\text{Dom } \mathcal{E} := \left\{ u : X \to \mathbb{R} \mid \lim_{n \to \infty} \mathcal{E}_n(u|_{D_n}) < \infty \right\}.$$ 

**Proposition 4.3.** $(\mathcal{E}, \text{Dom } \mathcal{E})$ is a resistance form.

This follows directly from the above along with [Kig12, Theorem 3.13]. The domain of the resistance form $\mathcal{E}$ is a subset of the continuous functions on the metric space $\Omega$, which is the completion of $V_*$ with respect to $R$, the effective resistance metric of $\mathcal{E}$.

Consider as before $d_n$ to be the standard geodesic metric defined on the quantum graphs $X_n$. If we consider $X_n$ as embedded in $\mathbb{R}^2$, $d_n$ locally agrees with the Euclidean distance. In general $d_n(x, y) \geq d_{n+1}(x, y)$, and if $x, y \in J_n$, then $d_n(x, y) = d_{n+1}(x, y)$ for all $m > n$. This allows us to consider the limit $d_G := \lim_{n \to \infty} d_n$, and extend it to a metric on all of $X$.

**Proposition 4.4.** The metric $d_G$ induces the same topology as the Euclidean one. Further $d_G$ is self-similar on $X$ in that $d_G(F_w(x), F_w(y)) = (1 - \alpha)^k d_G(x, y)$ for any $w \in A^n$.

**Proof.** For any $k > 0$, there is a bijection between paths in $X_k$ from $x$ to $y$, and paths in $F_w(X_k) \subset X_{k+n}$. It is easy to see that a minimizing path will not leave $F_w(X_k)$, and so this implies that $d_{n+k}(F_w(x), F_w(y)) = (1 - \alpha)^k d_k(x, y)$. Self-similarity follows by passing to the limit.

If $x$ is in the interior of $\Phi_e(I_e)$ for some $e \in J_n$, then clearly, for all $m \geq n$, $d_m$ is isometric to the Euclidean distance in a small enough region around $x$.

To complete the proof it suffices to show that if $x \in V_n$ is a corner of $F_w(X)$, $w \in A^k$, then $\sup_{y \in F_w(X)} d_G(x, y) = C(1 - \alpha)^k$ where $C \in (0, \infty)$ is constant.

In fact, we can take $C = \sup_{y \in X} d_G(x, y)$ where $x = p_1$ is a corner of the original triangle. In this case $C = 1 + \alpha/2$, which is attained where $y = p_4$ is the center of the opposite side. Clearly, the maximizing $y$ cannot be in $F_1(X), F_5(X)$ or $F_4(X)$.

---

**Figure 4.** Reduction of the first level approximation network of the Hanoi attractor.
To see that $y$ is not in $F_2(X)$ or $F_3(X)$, assume to the contrary that $y \in F_2(X)$. In this case we see that a minimizing path must pass through $F_2(p_1)$ — but the minimizing path from $p_1$ to $F_2(p_3)$ passes through $F_2(p_1)$. Thus,

$$C = d_G(p_1, F_2(p_1)) + \sup_{z \in F_2(X)} d_G(F_2(p_1), z),$$

which implies that $y$ maximizes $d_G(F_2(p_1), z)$ and in particular $F_2(y) = y$. This means that $y = p_2$, which is a contradiction because $d_G(p_1, p_2) = 1 \leq 1 + \alpha/2$. \hfill \Box

Since $d_G$ is approximated by $d_n$, one can show that $d_G$ is a geodesic metric using an approximate midpoint argument. Further, it can be seen that the length structure induced by $d_G$ is the same as the length metric induced by the restriction of the Euclidean distance.

To show that $\mathcal{E}$ is a resistance form on the space $X$, we need the following theorem

**Theorem 4.5.** The space $\Omega$ is homeomorphic to $X$, where $X$ is given the subspace topology from $\mathbb{R}^2$. In particular, $\mathcal{E}$ is a resistance form with $\text{Dom} \mathcal{E} \subset C(X)$.

**Proof.** As we saw in the proof of Lemma 4.1, we know that if $x, y$ belong to the same edge of $J_n$, then $R(x, y) \leq |x - y|$.

It can be seen that $R(x, y) \geq |x - y|/3$ as follows: let $n$ be the smallest level such that $x$ and $y$ are in $X_n$. Since we are still assuming that $x$ and $y$ are on the same edge, call that edge $e$. Then $e$ is adjacent to $F_w(X)$ for some $w \in A^n$, assuming without loss of generality that $y$ is closer to $F_w(X)$ than $x$ is. Because $n$ is the smallest such that $e \in J_n$, $e$ is the shortest such edge.

This allows us to construct a function $u$ with $u(x) = 0, u(y) = 1$, interpolating linearly between $x$ and $y$ and staying constant outside. Moreover, $u|_{F_w(X)} \equiv 1$, and it linearly decays from 1 to 0 on the other (at most two) edges adjacent to $F_w(X)$. Finally, let $u$ be constant zero everywhere else. Thus $\mathcal{E}(u) \leq 3/|x - y|$ which implies that $R(x, y) \geq |x - y|/3$.

We can conclude, for $x$ in the interior of $I_e$, that neighbourhoods around $x$ with respect to $R$ are the same as those with respect to $d$ (and $d_G$).

If $x \in V_n$, where $n$ is the minimal such $n$, then $x \in F_w(X)$ for some $w \in A^n$, and it is adjacent to an edge $e$. Fixing an $r > 0$, it follows from Lemma 4.1 that $B_{d_G}(x, r) \subset B_R(x, r)$, so we must show that there is a resistance ball contained inside $B_d(x, r)$. We know that $F_v(X) \subset B_d(x, r)$ for some $v \in A^m$ such that $x \in F_v(X)$.

In the proof of Proposition 4.2, we saw that the diameter of $F_v(X)$ with respect to $R$ is less than $\left(\frac{3(1-\alpha)}{5}\right)^m$ and in particular, $B_R(x, r_0) \setminus e \subset F_v(X)$ for $r_0 < \left(\frac{3(10\alpha)}{5}\right)^m$. 

\[\]
Because of this and the above argument, if we take
\[ r_0 = \min \left\{ \left( \frac{3(1 - \alpha)}{5} \right)^m r \right\} \]
then \( B_{r_0}(x) \subseteq B_d(x, r) \). \qed

5. Spectral asymptotics

It is explained in [Kig12, Chapter 9] that a resistance form together with a locally finite regular measure induces a Dirichlet form on the corresponding \( L^2 \)-space. Thus introducing an appropriate measure \( \mu \) on \( X \), we can get a Dirichlet form and therefore a Laplacian on \( L^2(X, \mu) \). The spectral properties of this operator strongly depend on the measure, that we choose in a weakly self-similar manner in view of the geometric properties of \( X \).

Recall from the introduction the parameters \( 0 < \beta < \frac{1}{3} \) and \( s = \frac{1 - 3\beta}{3} \). For each \( w \in A^* \) we define
\[ \mu(F_w(X)) := s^{|w|} \]
and notice that \( \beta \) and \( s \) are related in such a way that \( \mu(X) = 1 \) and \( \mu(I_e) = \beta \) for each \( e \in J_1 \). The condition \( 0 < \beta < \frac{1}{3} \) is needed for technical reasons that have already been discussed in the introduction.

As a direct consequence of Theorem 4.5, \( X \) is compact with respect to the resistance metric, thus it follows from [Kig12, Corollary 6.4] that the induced Dirichlet form coincides with \((\mathcal{E}, \text{Dom } \mathcal{E})\). Next definition is a well-known fact from the theory of Dirichlet forms that can be found in [FOT11, Corollary 1.3.1].

**Definition 5.1.** The Laplacian associated with \((\mathcal{E}, \text{Dom } \mathcal{E})\) is the unique non-negative self-adjoint operator \( \Delta_\mu : \text{Dom } \Delta_\mu \to L^2(X, \mu) \) such that \( \text{Dom } \Delta_\mu \) is dense in \( L^2(X, \mu) \) and
\[ \mathcal{E}(u, v) = -\int_X \Delta_\mu u \cdot v \, d\mu \quad \forall v \in \text{Dom } \mathcal{E}. \]

From now on, we will denote by \( r := \frac{1-\alpha}{2} \) the scaling factor of the similitudes \( F_1, F_2, F_3 \) and write \( I_e = [0, r^n \alpha] \) for any \( e \in J_{n+1} \setminus J_n, n \in \mathbb{N}_0 \).

**Lemma 5.1.** For any \( u \in \text{Dom } \mathcal{E} \),
\[ \mathcal{E}(u, u) = \sum_{i=1}^3 r^{-1} \mathcal{E}(u \circ F_i, u \circ F_i) + \sum_{e \in J_1} \int_0^\alpha |u'|^2 \, dx. \]

**Proof.** Let \( n \in \mathbb{N}_0 \) and \( u \in \mathcal{F}_n \).
\[ \mathcal{E}_n(u, u) = \sum_{k=1}^n \sum_{e \in J_k \setminus J_{k-1}} \int_0^{r^{k-1}\alpha} |u'|^2 \, dx \]
\[ = \sum_{i=1}^3 \sum_{k=1}^{n-1} \sum_{e \in F_i(J_k \setminus J_{k-1})} \int_0^{r^k \alpha} |u'|^2 \, dx + \sum_{e \in J_1} \int_0^{r^{k-1}\alpha} |u'|^2 \, dx \]
Applying the transformation of variables $x = F_i(y)$ we get that
\[
\mathcal{E}_n(u, u) = \sum_{i=1}^{3} \sum_{k=1}^{n-1} \sum_{e \in F_i(J_{k-1})} r^{-1} \int_0^{r^{k-1}\alpha} |(u \circ F_i)'|^2 dy + \sum_{e \in J_1} \int_0^{\alpha} |u'|^2 dx
\]
\[
= \sum_{i=1}^{3} r^{-1} \mathcal{E}_{n-1}(u \circ F_i, u \circ F_i) + \sum_{e \in J_1} \int_0^{\alpha} |u'|^2 dx.
\]
Now, letting $n \to \infty$ in both sides of the equality proves the assertion. \qed

By induction we get the following generalization of this Lemma.

**Corollary 5.2.** For any $u \in \text{Dom} \mathcal{E}$ and $m \in \mathbb{N}$,
\[
\mathcal{E}(u, u) = \sum_{w \in \mathcal{A}^m} r^{-m} \mathcal{E}(u \circ F_w, u \circ F_w) + \sum_{k=0}^{m-1} s^{-k} \sum_{w \in \mathcal{A}^k} \sum_{e \in J_1} \int_0^{\alpha} |(u \circ F_w)'|^2 dx.
\]

Let us now introduce the notation: Given two functions $f, g : \mathbb{R} \to \mathbb{R}$, we write $f(x) \asymp g(x)$ when there exist constants $c_1, c_2 > 0$ such that
\[
c_1 g(x) \leq f(x) \leq c_2 g(x).
\]

Recall that the eigenvalue counting function of $\Delta_\mu$ subject to Neumann (resp. Dirichlet) boundary conditions is defined as
\[
N_N(x) := \#\{\lambda \text{ Neumann eigenvalue of } \Delta_\mu \mid \lambda \leq x\},
\]
respectively
\[
N_D(x) := \#\{\lambda \text{ Dirichlet eigenvalue of } \Delta_\mu \mid \lambda \leq x\}
\]
counted with multiplicity.

This function can also be defined for Dirichlet forms by considering that $\lambda \in \mathbb{R}$ is an eigenvalue of $\mathcal{E}$ if and only if there exists $u \in \text{Dom} \mathcal{E}$ such that $\mathcal{E}(u, v) = \lambda \int_{K_\alpha} u v \, d\mu$ for all $v \in \text{Dom} \mathcal{E}$. In this case the eigenvalue counting function
\[
N(x; \mathcal{E}, \text{Dom} \mathcal{E}) := \#\{\lambda \text{ eigenvalue of } \mathcal{E}, \lambda \leq x\}
\]
coincides with $N_N(x)$ (see [Lap91, Proposition 4.1]). Analogously it holds that
\[
N_D(x) = N(x; \mathcal{E}^0, \text{Dom} \mathcal{E}^0),
\]
where $\text{Dom} \mathcal{E}^0 := \{u \in \text{Dom} \mathcal{E} \mid u|_{V_0} \equiv 0\}$ and $\mathcal{E}^0 := \mathcal{E}|_{\text{Dom} \mathcal{E}^0 \times \text{Dom} \mathcal{E}^0}$.

The asymptotic behaviour of the eigenvalue counting function is described by what we call here the **spectral dimension** of $X$, that is the non-negative number $d_S$ such that
\[
N_{N/D}(x) \asymp x^{d_S/2}.
\]
The expression $N_{N/D}(x)$ means that the assertion holds for both $N_N(x)$ and $N_D(x)$ and we will use it in the following to simplify notation.

The main result of this section is the following theorem.
Theorem 5.3. Let $s, r > 0$ be the quantities defined at the beginning of this section. Then we have that
\[
d_S X = \begin{cases} 
1, & 0 < rs \leq \frac{1}{9}, \\
-\log \frac{9}{\log(rs)}, & \frac{1}{9} < rs < \frac{1}{6}.
\end{cases}
\]
This result is a consequence of the proposition below.

Proposition 5.4. There exist constants $C_1, C_2 > 0$ and $x_0 > 0$ such that
(i) if $0 < rs < \frac{1}{9}$, then
\[
C_1 x^{\frac{1}{2}} \leq N_D(x) \leq N_N(x) \leq C_2 x^{\frac{1}{2}},
\]
(ii) if $rs = \frac{1}{9}$, then
\[
C_1 x^{\frac{1}{2}} \log x \leq N_D(x) \leq N_N(x) \leq C_2 x^{\frac{1}{2}} \log x,
\]
(iii) if $\frac{1}{9} < rs < \frac{1}{6}$, then
\[
C_1 x^{-\frac{\log 3}{\log(rs)}} \leq N_D(x) \leq N_N(x) \leq C_2 x^{-\frac{\log 3}{\log(rs)}}
\]
for all $x > x_0$.

The proof of this proposition is divided into several lemmas that estimate the eigenvalue counting function $N_N(x)$ and $N_D(x)$ and it mainly follows the ideas of [Kaj10], which can be applied due to the choice of the measure $\mu$.

In the following, we set
\[
\|u\|_{\mathcal{E}(1)} := \left( \mathcal{E}(u, u) + \|u\|^2_{L^2(X, \mu)} \right)^{1/2}
\]
which is a norm on $\text{Dom } \mathcal{E}$.

Upper bound. Let us write $X_w := F_w(X)$ for each $w \in \mathcal{A}^*$ and define $X_m := \bigcup_{w \in \mathcal{A}^m} X_w$ and $I_m := X \setminus X_m$ for each $m \in \mathbb{N}$.

On the one hand, we consider the pair $(\mathcal{E}_{I_m}, \text{Dom } \mathcal{E}_{I_m})$ given by
\[
\text{Dom } \mathcal{E}_{I_m} := \{ u \in \text{Dom } \mathcal{E} | \text{supp}(u) \subseteq I_m \},
\]
\[
\mathcal{E}_{I_m} := \mathcal{E}|_{\text{Dom } \mathcal{E}_{I_m} \times \text{Dom } \mathcal{E}_{I_m}},
\]
where the closure is taken with respect to $\|\cdot\|_{\mathcal{E}(1)}$. Since $I_m$ is an open set, we know by [FOT11, Theorem 4.4.3] that this is a Dirichlet form on $L^2(I_m, \mu|_{I_m})$. This space can also be identified with $\bigoplus_{e \in I_m} L^2(I_e, \mu|_{I_e})$ and $\text{Dom } \mathcal{E}_{I_m}$ can be identified with $\bigoplus_{e \in I_m} H^1_0(I_e, \mu|_{I_e})$.

On the other hand, we take $\text{Dom } \mathcal{E}_{X_m} := (\text{Dom } \mathcal{E}_{I_m})^\perp$ the orthogonal complement on $\text{Dom } \mathcal{E}$ with respect to $\mathcal{E}(1)$ and $\mathcal{E}_{X_m} := \mathcal{E}|_{\text{Dom } \mathcal{E}_{X_m} \times \text{Dom } \mathcal{E}_{X_m}}$. The pair $(\mathcal{E}_{X_m}, \text{Dom } \mathcal{E}_{X_m})$ can be regarded as a (regular) Dirichlet form on $L^2(X_m, \mu|_{X_m})$ since $\text{Dom } \mathcal{E}_{X_m}$ is a dense subspace of $L^2(X_m, \mu|_{X_m})$. $(\text{Dom } \mathcal{E}_{X_m}, \mathcal{E}_{X_m} + \|\cdot\|^2_{L^2(X_m, \mu|_{X_m})})$ is complete because $\text{Dom } \mathcal{E}_{X_m}$ closed with respect to $\|\cdot\|_{\mathcal{E}(1)}$ and the Markov property is inherited from $(\mathcal{E}, \text{Dom } \mathcal{E})$. 
Lemma 5.5. For each $m \in \mathbb{N}$
\[ N_N(x) \leq N(x; \delta_{X_m}, \text{Dom} \delta_{X_m}) + N(x; \delta_{I_m}, \text{Dom} \delta_{I_m}) \quad \forall x \geq 0. \]

Proof. Since $\text{Dom} \delta \subseteq \text{Dom} \delta_{X_m} \oplus \text{Dom} \delta_{I_m}$, the minimax principle yields $N_N(x) = N(x; \delta, \text{Dom} \delta) \leq N(x; \delta, \text{Dom} \delta_{X_m} \oplus \text{Dom} \delta_{I_m})$. Moreover, $\delta = \delta_{X_m} \oplus \delta_{I_m}$, hence \[ N(x; \delta_{X_m} \oplus \delta_{I_m}, \text{Dom} \delta_{X_m} \oplus \text{Dom} \delta_{I_m}) = N(x; \delta_{X_m}, \text{Dom} \delta_{X_m}) + N(x; \delta_{I_m}, \text{Dom} \delta_{I_m}) \]
holds by [Lap91, Lemma 4.2] and we are done. \qed

Remark 3. Note that in this proof we first consider $\delta_{X_m}$ and $\delta_{I_m}$ as bilinear forms in $L^2(X, \mu)$ while the last step consider each of them on $L^2(X_m, \mu|_{X_m})$ and $L^2(I_m, \mu|_{I_m})$ respectively.

Lemma 5.6. For each $m \in \mathbb{N}$, define
\[ \lambda(L) := \sup \{ \delta_{X_m}(u, u) \mid u \in L, \|u\|_{L^2(X_m, \mu|_{X_m})} = 1 \}, \quad L \subseteq \text{Dom} \delta_{X_m} \text{ subspace}, \]
\[ \lambda_n := \inf \{ \lambda(L) \mid L \subseteq \text{Dom} \delta_{X_m}, \dim L = n \}. \]
Then, it holds that
\[ \lambda_{3^m+1} \geq C_P (rs)^{-m}. \]

Proof. By Corollary 5.2 and the definition of $\delta_{X_m}$ we have that
\[ \delta(u, u) = \sum_{w \in A^m} r^{-m} \delta(u \circ F_w, u \circ F_w) + \sum_{k=0}^{m-1} \sum_{w \in A^k} \sum_{e \in E} \int_0^\infty |(u \circ F_w)|^2 \ dx \]
for all $u \in \text{Dom} \delta_{X_m}$, note that all of the components of the above sum are positive.

Now, we follow the same argument as in [Kaj10, Lemma 4.5], which we include for completeness: consider $L_0 := \{ \sum_{w \in A^m} a_w 1_{X_m} \mid a_w \in \mathbb{R} \}$. This is a $3^m$-dimensional subspace of $\text{Dom} \delta_{X_m}$ and $\delta_{X_m}|_{L_0 \times L_0} \equiv 0$. Now, given a $(3^m + 1)$-dimensional subspace $L \subseteq \text{Dom} \delta_{X_m}$, we consider $\bar{L} := L_0 + L$, that is a finite-dimensional subspace of $\text{Dom} \delta_{X_m}$. The non-negative self-adjoint operator associated with $\delta|_{L \times \bar{L}}$ may be expressed by a matrix $A$ whose $3^m + 1$-th smallest eigenvalue is given by
\[ \lambda_A := \inf \{ \lambda(L') \mid L' \subseteq \bar{L}, \dim L' = 3^m + 1 \}. \]
Let us call $u_A$ its corresponding eigenfunction and renormalise it, so that $\int_{X_m} u_A^2 \ d\mu = 1$.

Since $(\delta, \text{Dom} \delta)$ is a resistance form on $X$, the associated resistance metric $R$ is compatible with the original topology of $X$ by Theorem 4.5, and $u_A$ is orthogonal to $L_0$, the uniform Poincaré inequality (see [Kaj10, Definition 4.2]) holds for $u_A$. This together with equality (5.1) leads to
\[ \lambda(L) \geq \lambda_A = \delta_{X_m}(u_A, u_A) \geq \sum_{w \in A^m} r^{-m} \delta(u_A \circ F_w, u_A \circ F_w) \]
\[ \geq C_P \sum_{w \in A^m} \frac{r^{-m}}{\mu(X_w)} \int_{X_w} |u_A|^2 \ d\mu \]
\[ \geq \frac{r^{-m}C_P}{\max_{w \in A^m} \mu(X_w)} \sum_{w \in A^m} \int_{X_w} |u_A|^2 \ d\mu \geq C_P \left( \frac{1}{rs} \right)^m, \]
where $C_P > 0$ is the constant of the uniform Poincaré inequality. \qed

Lemma 5.7. There exist a constant $\tilde{C} > 0$ and $x_0 > 0$ such that
(i) if $0 < rs < \frac{1}{6}$, then
\[ N_N(x) \leq \tilde{C} x^{1/2}, \]

(ii) if $rs = \frac{1}{6}$, then
\[ N_N(x) \leq \tilde{C} x^{1/2} \log x, \]

(iii) if $\frac{1}{6} < rs < \frac{1}{5}$, then
\[ N_N(x) \leq \tilde{C} x^{\frac{\log 3}{\log(rs)}}, \]

for all $x \geq x_0$.

**Proof.** Let $x_0 := \frac{4\pi^2}{\alpha \beta rs} > 0$. For any $x \geq x_0$ we can find $m > 0$ such that
\[ \frac{4\pi^2}{\alpha \beta (rs)^m} \leq x < \frac{4\pi^2}{\alpha \beta (rs)^{m+1}}. \]

By Lemma 5.6 we know that
\[ \lambda_{3m+1} \geq \frac{C_P}{(sr)^m} \geq x \]
and hence
\[ N(x; \mathcal{E}_{X_m}, \text{Dom } \mathcal{E}_{X_m}) \leq 3^m \leq \tilde{C} x^{\frac{\log 3}{m \log(rs)}}, \]

for $\tilde{C} = 3C_P^{\frac{\log 3}{m \log(rs)}}$.

On the other hand, notice that $I_m$ is the disjoint union of 1-dimensional intervals, hence
\[ N(x; \mathcal{E}_{I_m}, \text{Dom } \mathcal{E}_{I_m}) = \sum_{e \in J_m} N_e(x), \]

where $N_e(x)$ denotes the eigenvalue counting function of the Laplacian on $L^2(I_e, \mu_{|I_e})$, that we denote by $\Delta_{\mu_{|I_e}}$.

Without loss of generality let us consider $I_e = [0, s^k \alpha]$. Suppose that $\lambda$ is an eigenvalue of the Laplacian $\Delta_{\mu_{|I_e}}$ with eigenfunction $f \in H^1_0(I_e, \mu_{|I_e})$. Then,
\[ \int_0^{s^k \alpha} f'g' \, dx = \lambda \int_0^{s^k \alpha} fg \, d\mu = \frac{\lambda \mu(I_e)}{m(I_e)} \int_0^{s^k \alpha} fg \, dx \]
for all $g \in H^1_0(I_e)$, where $m(I_e)$ denotes the Lebesgue measure of $I_e$. This means, $\lambda \mu(I_e)/m(I_e)$ is an eigenvalue of the classical Laplacian $\Delta$ on $L^2(I_e, dx)$ subject to Dirichlet boundary conditions. The converse holds by the same calculation, so we can say that $N_e(x) = N^I_D(\frac{\mu(I_e)}{m(I_e)})$ for all $x \geq 0$. Here $N^I_D(\cdot)$ denotes the eigenvalue counting function of the classical Laplacian on $L^2(I_e, dx)$ subject to Dirichlet boundary conditions.

From Weyl’s theorem for the asymptotics of the eigenvalue counting function for the classical Laplacian on bounded sets of $\mathbb{R}$ (see [Wey12]), we know that
\[ N_e(x) \asymp \frac{(\mu(I_e)m(I_e))^{1/2}}{2\pi} x^{1/2} = \frac{(\alpha \beta)^{1/2}(rs)^{m/2}}{2\pi} x^{1/2}, \]

hence
\[ N(x; \mathcal{E}_{I_m}, \text{Dom } \mathcal{E}_{I_m}) \asymp \sum_{n=1}^m \frac{(\alpha \beta)^{1/2}(9rs)^{n/2}}{2\pi} x^{1/2}, \]

for $\tilde{C} = 3C_P^{\frac{\log 3}{m \log(rs)}}$. 
which is nothing but the counting function of the set
\[ \bigcup_{n=0}^{m} \left\{ \frac{(2\pi k)^2}{\alpha \beta (rs)^n} \mid k = 1, 2, \ldots \right\}. \]

Now, note that (5.2) is equivalent to
\[ m \leq \frac{\log x}{-\log(rs)} + C < m + 1, \]
where
\[ C = \frac{\log(\alpha \beta) - 2 \log(2\pi)}{-\log(rs)} \]
and so we have that
\[ \sum_{n=1}^{m} (9rs)^{n/2} \asymp \int_{0}^{\log x - \log(rs) + C} (9rs)^{\frac{x}{2}} dx. \]

If \( 0 < rs < 1/9 \), this integral is bounded by a constant and we get from (5.4) that
\[ N(x; \mathcal{E}_m', \text{Dom} \mathcal{E}_m') \asymp x^{\frac{1}{2}}. \]

If \( rs = \frac{1}{9} \), then \( 9rs = 1 \) and the integral becomes \( \frac{\log x}{\log(rs)} + C \). Moreover, \( \frac{\ln 3}{\ln(rs)} = \frac{1}{2} \) and hence (5.3) and (5.4) lead to
\[ N(x; \mathcal{E}_m', \text{Dom} \mathcal{E}_m') \asymp x^{\frac{1}{2}} \log x. \]

Finally, if \( \frac{1}{9} < rs < \frac{1}{6} \), then we have that
\[ \sum_{n=1}^{m} (9rs)^{n/2} \asymp \int_{0}^{\log x - \log(rs) + C} (9rs)^{\frac{x}{2}} dx \]
\[ = \frac{2}{\log(9rs)} \left[ (9rs)^{\frac{x}{2}} \right]_{0}^{\log x - \log(rs) + C} \]
\[ = \frac{2}{\log(9rs)} \left( (9rs)^{\frac{\log 9}{2} \log(rs)} - 1 \right), \]
hence
\[ N(x; \mathcal{E}_m', \text{Dom} \mathcal{E}_m') \asymp x^{\frac{\log(9rs)}{-2 \log(rs)} + \frac{1}{2}} \]
and since
\[ \frac{\log(9rs)}{-2 \log(rs)} + \frac{1}{2} = \frac{\log(rs) - \log(9rs)}{2 \log(rs)} = \frac{\log 9}{-2 \log(rs)}, \]
we get
\[ N(x; \mathcal{E}_m', \text{Dom} \mathcal{E}_m') \asymp x^{\frac{\log 3}{-2 \log(rs)}}. \]

The assertion now follows from 5.3 and Lemma 5.8. \( \square \)
**Lower bound.** Recall that \((\mathcal{E}^0, \text{Dom } \mathcal{E}^0)\) is the Dirichlet form whose associated non-negative self-adjoint operator is the Laplacian \(\Delta_\mu\) subject to Dirichlet boundary conditions. Let us now write for each \(w \in \mathcal{A}^m\) and \(m \in \mathbb{N}\), \(X^0_m \coloneqq F_w(\mathcal{X} \setminus \mathcal{V}_0)\) and \(X^0_m \coloneqq \bigcup_{w \in \mathcal{A}^m} X^0_m\). Since \(X^0_m\) is open, we know by [FOT11, Theorem 4.4.3] that the pair \((\mathcal{E}^0_{X^0_m}, \text{Dom } \mathcal{E}^0_{X^0_m})\) given by

\[
\begin{align*}
\text{Dom } \mathcal{E}^0_{X^0_m} & := \{ u \in \text{Dom } \mathcal{E} | \text{supp}(u) \subseteq X^0_m \}, \\
\mathcal{E}^0_{X^0_m} & := \mathcal{E}|_{\text{Dom } \mathcal{E}^0_{X^0_m} \times \text{Dom } \mathcal{E}^0_{X^0_m}},
\end{align*}
\]

where the closure is taken with respect to \(||\cdot||_{\mathcal{E}^{(1)}}\), is a Dirichlet form on \(L^2(X^0_m, \mu_{|X^0_m})\).

Analogously, we define for each \(w \in \mathcal{A}^m\) the Dirichlet form \((\mathcal{E}_{X^0_w}, \text{Dom } \mathcal{E}_{X^0_w})\) on \(L^2(X^0_w, \mu_{|X^0_w})\).

**Lemma 5.8.** For each \(m \in \mathbb{N}\) and \(x \geq 0\),

\[
N_D(x) \geq \sum_{w \in \mathcal{A}^m} N(x; \mathcal{E}_{X^0_w}, \text{Dom } \mathcal{E}_{X^0_w}) + N(x; \mathcal{E}_{X^m}, \text{Dom } \mathcal{E}_{X^m}).
\]

**Proof.** The proof is completely analogous to [Kaj10, Lemma 4.8]. \(\square\)

**Lemma 5.9.** For any \(m \in \mathbb{N}\) there exists \(C_D > 0\) such that

\[
\lambda_1(X^0_m) := \inf_{u \in \text{Dom } \mathcal{E}_{X^0_m}, u \neq 0} \frac{\mathcal{E}_{X^0_m}(u, u)}{\|u\|^2_{L^2(X^0_m, \mu_{|X^0_m})}} \leq \frac{C_D}{(sr)^m}
\]

for all \(w \in \mathcal{A}^m\).

**Proof.** Consider \(\nu \in \mathcal{A}^m\) such that \(X^0_w \subseteq X^0_m\). Since \(X^0_m\) is open and \(X^0_w\) compact, we know that there exists a function \(u \in \text{Dom } \mathcal{E}_{X^0_m}\) such that \(u_{|X^0_w} \equiv 1\), \(u \geq 0\) and \(\text{supp}(u) \subseteq X^0_m\).

Define

\[
uw(x) := \begin{cases} 
  u \circ F_w^{-1}(x), & x \in X^0_w, \\
  0, & x \in X^0_m \setminus X^0_w. 
\end{cases}
\]

Clearly \(\nuw \in \text{Dom } \mathcal{E}_{X^0_m}\) and analogously to the proof of Lemma 5.6 we have by Corollary 5.2 that

\[
\mathcal{E}_{X^0_m}(\nuw, \nuw) = \sum_{w \in \mathcal{A}^m} r^{-m} \mathcal{E}(\nuw \circ F_w, \nuw \circ F_w) + \sum_{k=0}^{m-1} \sum_{w \in \mathcal{A}^k} \sum_{r \in J_k} \int_0^s |(\nuw \circ F_w)|^2 \, dx.
\]

Since \(\text{supp}(\nuw) \subseteq X^0_m\), the last term of this sum equals zero and the definition of \(\nuw\) leads to

\[
\mathcal{E}_{X^0_m}(\nuw, \nuw) = r^{-m} \mathcal{E}(\nuw \circ F_w, \nuw \circ F_w) = r^{-m} \mathcal{E}(u, u).
\]

On the other hand, by definition of \(\mu\) we have that

\[
\int_{X^0_w} |\nuw|^2 \, d\mu(x) = \int_{X^0_w} |\nuw|^2 \, d\mu(F_w(y)) \geq \int_{X^0_w} |u|^2 \, d\mu(F_w(y)) = \mu(F_w(X^0_w)) \geq s|w| \mu(X_w)
\]

(5.6)
Finally, applying (5.5) and (5.6) we obtain

$$\lambda_1(X_0^w) \leq \frac{\mathcal{E}_{X_0^w}(u^w, u^w)}{\|u^w\|_{L^2(X_0^w; \mu_{X_0^w})}^2} \leq \frac{r^{-m} \mathcal{E}(u, u)}{s^{\|v\|} \mu(X_0^w)} = \frac{C_D}{(rs)^m},$$

where $C_D := \frac{\mathcal{E}(u, u)}{r^{\|v\|}}$ is independent of $w$ and the assertion follows. \hfill \Box

**Lemma 5.10.** There exist a constant $C' > 0$ and $x_0 > 0$ such that

(i) if $0 < rs < \frac{1}{9}$, then

$$C' x^{\frac{1}{2}} \leq N_D(x),$$

(ii) if $rs = \frac{1}{9}$, then

$$C' x^{\frac{1}{2}} \log x \leq N_D(x),$$

(iii) if $\frac{1}{9} < rs < \frac{1}{6}$, then

$$C' x^{\log 3 - \log(rs)} \leq N_D(x)$$

for all $x \geq x_0$.

**Proof.** Analogously to the proof of Lemma 5.7, let $x_0 := \frac{4\pi^2}{\alpha \beta rs} > 0$ and consider $x \geq x_0$. Then there exists $m > 0$ such that

$$\frac{4\pi^2}{\alpha \beta (rs)^m} \leq x < \frac{4\pi^2}{\alpha \beta (rs)^{m+1}}.\tag{5.7}$$

By Lemma 5.9, we have that

$$\lambda_1(X_0^w) \leq \frac{C_D}{(sr)^m} \leq x$$

and hence $N(x; \mathcal{E}_{X_0^w}, \text{Dom} \ \mathcal{E}_{X_0^w}) \geq 1$ for all $w \in \mathcal{A}^m$. It follows from Lemma 5.8 that

$$N_D(x) \geq C' x^{\log 3 / \log(rs)}$$

for $C' = \frac{1}{3} \frac{\log 3}{\log(rs)}$.

Now, the very same argument from Lemma 5.7 together with Lemma 5.8 completes the proof. \hfill \Box

**Proof of Theorem 5.4.** Since $\text{Dom} \mathcal{E}^0 \subseteq \text{Dom} \mathcal{E}$ and $\mathcal{E}^0 = \mathcal{E}|_{\text{Dom} \mathcal{E}^0 \times \text{Dom} \mathcal{E}^0}$, the min-max principle yields $N_D(x) \leq N_N(x)$ for all $x > 0$. The statement follows directly from Lemmas 5.7 and 5.10. \hfill \Box

Notice that this result gives us the spectral dimension of $X$,

$$d_s X = \left\{ \begin{array}{ll} 1, & 0 < rs \leq \frac{1}{9}, \\ \frac{\log 9}{\log(rs)}, & \frac{1}{9} < rs < \frac{1}{6}, \end{array} \right.$$  

as stated in Theorem 5.3.
6. Fractal quantum graphs

**Definition 6.1.** A separable compact connected locally connected metric space \((X, d)\) is called a fractal quantum graph if there is a sequence of lengths \(\{\ell_k\}_{k=1}^\infty\) in \((0, \infty)\) and maps \(\Phi_k : [0, \ell_k] \to X\) which are local isometries and homeomorphisms between \([0, \ell_k]\) and \(\Phi_k([0, \ell_k])\). In addition, assume that \(\Phi_j((0, \ell_j)) \cap \Phi_k((0, \ell_k)) = \emptyset\) for all \(j \neq k\) and that the set
\[
X \setminus \bigcup_{k=1}^{\infty} \Phi_k((0, \ell_k))
\]
is a totally disconnected set.

Define \(\mathcal{F}_k\) to be the set of \(f \in C(X)\) such that \(f \circ \Phi_j \in H^1([0, \ell_j])\) for \(j \leq k\) and that \(f\) is locally constant away from \(\bigcup_{j=1}^k \Phi_j((0, \ell_k))\).

Define \(E(f) := \sum_{j=1}^k \int_{0}^{\ell_j} ((f \circ \Phi_j)'(x))^2 \, dx\) for \(f \in \mathcal{F}_k\). Consider the set
\[
D_k = \bigcup_{j=1}^k \{ \Phi_j(n\ell_j/2^k) \mid n = 0, 1, \ldots, 2^k \},
\]
and define \(\mathcal{E}_k(f) := \inf \{ E(g) \mid g \in \mathcal{F}_j, j \leq k, \text{ and } f|_{D_k} = g \}\) to be a quadratic form on \(\ell(D_k)\).

**Proposition 6.1.** For all \(n \geq 0\), \(\mathcal{E}_n\) is a resistance form, and \(\text{Tr}_{\Omega, n} \mathcal{E}_{n+1} = \mathcal{E}_n\). Thus \(D_n\) is a compatible system.

**Proof.** This follows in a similar vein as the first proof of Proposition 4.2. The most notable difference is that to prove that \(\mathcal{E}_n \geq 0\), we observe that if \(g \in \ell(D_n)\) and \(f \in \mathcal{F}_j\) is such that \(f|_{D_n} = g\) is non-constant, it must be non-constant on some image of \(\Phi_1([0, \ell_1])\), and thus \(E(f) \geq \int_{0}^{\ell_1} ((f \circ \Phi_1)'(x))^2 \, dx \geq \int_{0}^{\ell_1} h(x) \, dx\). Where \(h\) is the function which linearly interpolates between the values of \(g \circ \Phi_1\) on \(D_n\) and is independent of \(j\). □

Now applying the results from [Kig12, Theorem 3.13], we get the following theorem.

**Theorem 6.2.** There is a resistance form \(\mathcal{E}\) on \(\Omega\) such that \(\text{Tr}_{|D_n} \mathcal{E} = \mathcal{E}_n\) and \(\Omega\) is the completion of \(D_n\) with respect to the effective resistance metric \(R\).

It is important to note that without further assumptions on \(X\) we do not know in general that \(\Omega\) is homeomorphic to \(X\).

We define a sequence of pseudo-metrics \(R_n\) on \(X\) by
\[
R_n(x, y) := \sup \left\{ \frac{|f(x) - f(y)|^2}{E(f)} \mid f \in \mathcal{F}_n \right\}
\]
Notice that \(R_n(x, y) = 0\) if and only if \(x, y\) are in the same connected component of \((\bigcup_{i=1}^n \Phi_i((0, \ell_i)))^\circ\), and since \(\mathcal{F}_n \subset \mathcal{F}_{n+1}\), \(R_n(x, y) \leq R_{n+1}(x, y)\) for all \(x, y \in X\) and \(n\). Also, for \(x, y \in D_n\), \(R_n(x, y) \leq R(x, y)\). We now prove some properties of \(R_n\).

**Lemma 6.3.** For each \(n \geq 0\), \(R_n\) is a continuous function from \(X^2\) to \([0, \infty)\).
Proof. Consider the space $X/\sim$ where $\sim$ is the equivalence induced by the pseudometric $x \sim y$ if $R_n(x, y) = 0$. This space (with the metric induced by $R_n$) is isometric to the metric graph with edges $\{\Phi_i((0, \ell_i))\}_{i=1}^n$ and vertices corresponding to connected components of $\bigcup_{i=1}^n \Phi_i((0, \ell_i))$. Noting that the connected components of $\bigcup_{i=1}^n \Phi_i((0, \ell_i))$ are closed in $X$ with respect to $d$, it is easy to see that the projection from $X$ to $X/\sim$ is continuous with respect to $d$. \hfill $\Box$

Lemma 6.4. For all $x, y \in D_\ast$ there is $n \geq 0$ such that $R_m(x, y) > 0$ for all $m > n$.

Proof. It suffices to show that there is some $n$ such that $x, y$ are in different connected components of $\bigcup_{i=1}^n \Phi_i((0, \ell_i))$. Assuming to the contrary that for all $n$ the connected component $K_n \subset \bigcup_{i=1}^n \Phi_i((0, \ell_i))$ contains both $x$ and $y$. Since $K_n$ is closed, they are compact. Since they are connected and $\cap_{n=1}^\infty K_n$ is non-empty, this intersection is connected and contains both $x$ and $y$. This is a contradiction to our assumption that $\bigcup_{i=1}^\infty \Phi_i((0, \ell_i))$ is completely disconnected. \hfill $\Box$

In fact the above lemma is the reason why we have assumed that $\bigcup_{i=1}^\infty \Phi_i((0, \ell_i))$ is completely disconnected, so that the family $\cup_{n=1}^\infty \mathcal{F}_n$ separates points.

Proposition 6.5. There is an injective map $\phi : \Omega \to X$ such that $\phi|_{D_\ast}$ is the identity. In particular, if $\{x_n\}_{n=1}^\infty$ is an $R$-Cauchy sequence in $D_\ast$, then $\{x_n\}_{n=1}^\infty$ is a $d$-Cauchy sequence.

Proof. Because $X$ is compact with respect to $d$, any subsequence of $\{x_n\}$ must have a convergent subsequence, so to show that the sequence converges it suffices to show that every convergent subsequence of $\{x_n\}$ has the same limit.

Let $z_n, z'_n$ be subsequences of $x_n$ converging to $z, z'$ respectively. Then, $\lim_{n \to \infty} R_k(z_n, z'_n) = R_k(z, z') \leq \lim_{n \to \infty} R(z_n, z'_n) = 0$ for all $k$ and thus $z = z'$.

Since $\Omega$, the completion of $D_\ast$, with respect to $R$, can be realized as a quotient space of Cauchy sequences of $D_\ast$, this induces a map from $\phi : \Omega \to X$.

To see that this map is injective, consider two sequences $\{x_i\}, \{y_i\}$ in $D_\ast$ which are $R$-Cauchy, that is they converge to $x'$ and $y'$ in $\Omega$ respectively, such that, with respect to $d$ these sequences have the same limit, which we shall call $x$.

We shall show that $R_k$ is a family of uniformly bounded equicontinuous functions on the compact set $K^2$ where $K := \{x_i\} \cup \{y_i\} \cup \{x\}$ is considered as a subset of $X$ with the topology induced by $d$. From Arzela-Ascoli it follows that there is at least on accumulation point of $R_k$, but since pointwise $\lim_k R_k = R$ on a dense subset of $K^2$. It follows that $R(x', y') = \lim_k R_k(x, x) = 0$.

The family is uniformly bounded, because

$$R_k(x_i, y_i) \leq R(x_i, y_i) \leq R(x_i, x') + R(y_i, y') + R(x', y')$$

which is a bound which is independent of $k$ and $i$.

To see equicontinuity, notice that the only non-isolated points in $K^2$ have $x$ as one of the coordinates. Thus, we only have to show equicontinuity at points of the form $(x, z)$ where $z \in K$, and using the triangle inequality. We see,

$$|R_k(x, z) - R_k(x_n, z)| \leq R_k(x_n, x) \leq R(x_n, x'),$$

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or
\[ |R_k(x, z) - R_k(y_n, z)| \leq R_k(y_n, x) \leq R(y_n, y'). \]
In both strings the final inequality comes from the fact that
\[ R_k(x_n, x_m) \leq R_k(x_n, x_m) \]
for all \( k, n, m \) and take the limit as \( m \to \infty \) on both sides. Since the last term goes to 0 as \( n \to \infty \) independent of our choice on \( k \), this proves equicontinuity.

Now since we have shown that, for \( x, y \in \Phi_n((0, \ell_n)) \) then
\[ R(x, y) < d(x, y), \]
which implies the following

**Lemma 6.6.** \( \phi^{-1} \) is a homeomorphism when restricted to \( \Phi_n((0, \ell_n)) \).

In particular we can define \( \Psi_n := \phi^{-1} \circ \Phi_n : (0, \ell_n) \to \Omega \).

**Proposition 6.7.** If the length structure induced by \( d \), in the sense of [BBI01], is a length metric which agrees in topology with \( d \), in particular if \( d \) is a geodesic metric, then \( \Omega \) is homeomorphic to \( X \).

**Proof.** Without loss of generality, assume \( d \) is geodesic. It is elementary to show that
\[ d(x, y) \geq R_n(x, y) \]
for all \( x, y \in X \), this implies \( d(x, y) \geq R(x, y) \) for \( x, y \in D_\ast \). This implies that \( \phi \) is surjective, and hence bijective.

**Theorem 6.8.** The following are equivalent

1. \( \Omega \) is homeomorphic to \( X \).
2. \( \Omega \) is compact with respect to \( R \).
3. There is a sequence \( \varepsilon_n \), tending to 0 as \( n \to \infty \) such that \( R \)-diameter of each of the connected components of \( \Omega \setminus \bigcup_{i=1}^{n} \Psi_i((0, \ell_i)) \) is bounded by \( \varepsilon_n \).

**Proof.** Because \( X \) is compact, the (1) \( \Rightarrow \) (2) is trivial. For the (2) \( \Rightarrow \) (1), first notice that because \( \Omega \) is compact, for any \( d \)-convergent sequence \( \{x_i\} \subset D_\ast \), every subsequence must have a further subsequence which converges with respect to \( R \). These limits must be equal because \( \phi \) is injective. Thus \( \phi \) is a bijection and we can think of \( R(x, y) = R(\phi(x), \phi(y)) \) as a metric for \( x, y \in X \).

The same Arzela-Ascoli argument from the end of Proposition 6.5 yields that \( R \) is \( d \)-continuous on \( X^2 \), and thus \( \phi \) is continuous. The closed mapping theorem implies that \( \phi \) is a homeomorphism.

(3) \( \Leftrightarrow \) (2) follows because \( D_n \) is an \( \varepsilon_n \)-net, so \( \Omega \) is totally bounded (complete) and therefore compact. Conversely, if \( \Omega \) is compact, then we can find some finite set from \( D_\ast \) such that the \( R \)-balls of diameter \( \varepsilon \) cover \( \Omega \). Since this set is finite it must be contained in \( D_n \) for some \( n \), so \( D_n \), and in particular any connected component of \( \Omega \setminus \Psi_n((0, \ell_n)) \) is within \( \varepsilon \) of \( D_n \), and so the diameter is less than \( \varepsilon \). □

The sum \( \sum_{i=1}^{\infty} \ell_i \) provides a upper bound for effective resistance, we get the following.

**Corollary 6.9.** If \( \ell_i \) is a summable sequence then \( X \) is homeomorphic to \( \Omega \).
7. Generalized Hanoi-type quantum graphs

Let \( N_0 > 2 \) be a natural number and let \( \alpha > 0 \) be fixed. Further, consider the alphabet \( \mathcal{A}_{N_0} := \{1, \ldots, N_0\} \) and the contractions \( F_i: \mathbb{R}^{N_0-1} \to \mathbb{R}^{N_0-1}, \ i \in \mathcal{A}_{N_0} \). Each mapping \( F_i \) has contraction ratio \( r = \frac{1-\alpha}{2} (\leq 1) \) and fixed point \( p_i \). We also set \( V_{N_0} := \{p_1, \ldots, p_{N_0}\} \).

The generalized Hanoi attractor of parameters \( N_0 \) and \( \alpha \) is the unique non-empty compact subset of \( \mathbb{R}^{N_0-1} \) such that

\[
K_{\alpha,N_0} = \bigcup_{i=1}^{N_0} F_i(K_{\alpha,N_0}) \cup \bigcup_{\{i,j\} \subset \mathcal{A}_{N_0}} [i,j],
\]

where \([i,j]\) denotes the straight line joining the points \( F_i(p_j) \) and \( F_j(p_i) \) (note that \( i \neq j \)). It is easy to see that the Hausdorff dimension of this set is given by

\[
\max \left\{ N_0 - 2, \frac{\ln N_0}{\ln 2 - \ln(1-\alpha)} \right\}.
\]

Hence if we choose \( \alpha \) in the interval \((0, \frac{N_0-2}{N_0})\), then

\[
N_0 - 2 < \dim K_{\alpha,N_0} < N_0 - 1,
\]

and therefore a fractal. In the following, we will only consider \( \alpha \) belonging to this interval.

**Remark 4.** The case \( N_0 = 3 \) corresponds to the Hanoi attractor treated in Sections 3-5. In the case \( N_0 = 4 \), \( K_{\alpha,N_0} \) is fits into a tetrahedron of side length 1.

Let us now consider the generalized Hanoi attractor of parameter \( N_0 \) for a fixed \( \alpha \) and denote it by \( X_{N_0} \). This set may be approximated by the sequence of metric graphs \( (X_{N_0,n})_{n \in \mathbb{N}}, \) where \( X_{N_0,n} := (V_{N_0,n}, E_{N_0,n}, \partial) \) is defined analogously to Definition 3.1 just substituting \( \mathcal{A} \) by \( \mathcal{A}_{N_0} \).

By doing the obvious substitutions in Definition 3.2, we define the energy of the \( n \)-th approximation of \( X_{N_0}, \mathcal{E}_{N_0,n}: \mathcal{F}_{N_0,n} \times \mathcal{F}_{N_0,n} \to \mathbb{R} \) by

\[
\mathcal{E}_{N_0,n}(u, v) := \int_{X_{N_0}} u'v' \, dx
\]

for all \( u, v \in \mathcal{F}_{N_0,n} \), i.e. functions everywhere constant out of finitely many segments corresponding to “joining-type” edges of \( X_{N_0,n} \). By the same arguments as in Section 3 we get a suitable domain \( \text{Dom} \mathcal{E}_{N_0} \) on \( X_{N_0} \) such that

**Proposition 7.1.** \( (\mathcal{E}_{N_0}, \text{Dom} \mathcal{E}_{N_0}) \) is a resistance form.

In order to get a Dirichlet form from this resistance form, we need to define a measure \( \mu \) on \( X_{N_0} \). To this purpose we follow the construction of Section 5 and introduce a parameter \( \beta > 0 \) that measures the lines of length \( \alpha \). This needs to belong to the interval \( \left(0, \frac{2}{N_0(N_0-1)}\right)\) because otherwise,

\[
s := \mu(X'_{N_0}) = \frac{2 - N_0(N_0 - 1)\beta}{2N_0}
\]
would be zero or negative.

The definition of $s$ comes from the fact that we want the measure $\mu$ to satisfy

$$1 = \mu(X) = \frac{N_0(N_0 - 1)}{2} \beta + N_0 \mu(X'_{N_0}),$$

where $X'_{N_0}$ denotes any first-level copy of $X$ and $\frac{N_0(N_0 - 1)}{2}$ is the number of straight lines joining the different copies $X'_{N_0}$.

Following the proofs of Section 5 just replacing $X$ by $X_{N_0}$ and $(\mathcal{E}, \text{Dom} \mathcal{E})$ by $(\mathcal{E}_{N_0}, \text{Dom} \mathcal{E}_{N_0})$, one obtains the following statement on the spectral asymptotics of the corresponding eigenvalue counting function of the associated Laplacian.

**Proposition 7.2.** Let $r = 1 - \frac{\alpha}{2}$ and $s = \frac{2 - \beta N_0(N_0 - 1)}{2N_0}$. There exist constants $C_1, C_2 > 0$ and $x_0 > 0$ such that

(i) if $0 < rs < \frac{1}{N_0}$, then

$$C_1 x^{\frac{1}{2}} \leq N_D(x) \leq N_N(x) \leq C_2 x^{\frac{1}{2}},$$

(ii) if $rs = \frac{1}{N_0}$, then

$$C_1 x^{\frac{1}{2}} \log x \leq N_D(x) \leq N_N(x) \leq C_2 x^{\frac{1}{2}} \log x,$$

(iii) if $\frac{1}{N_0} < rs < \frac{1}{2N_0}$, then

$$C_1 x^{\frac{1}{2}} \frac{\log N_0}{-\log(rs)} \leq N_D(x) \leq N_N(x) \leq C_2 x^{\frac{1}{2}} \frac{\log N_0}{-\log(rs)}$$

for all $x > x_0$.

In this more general case, it follows directly from the choice of $\alpha$ and $\beta$ that

$$rs = \frac{(1 - \alpha)[2 - \beta N_0(N_0 - 1)]}{4N_0} < \frac{1}{2N_0}.$$ 

Finally, Proposition 7.2 leads to the spectral dimension of $X_{N_0}$.

**Theorem 7.3.** It holds that

$$d_S X_{N_0} = \begin{cases} 1, & 0 < rs \leq \frac{1}{N_0} \\ \frac{-\log N_0^2}{-\log(rs)}, & \frac{1}{N_0} < rs < \frac{1}{2N_0} \end{cases}$$

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