Glutty theories and the logic of antinomies

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Abstract

The logic of paradox (LP) [3, 13] is very well known in philosophy as a logic that accommodates ‘non-trivial glutty theories’ (theories that are negation inconsistent but don’t contain all sentences). We highlight a lesser known logic, namely, the logic of antinomies (LA) from Asenjo and Tamburino’s 1975 article [5], and compare it with LP. We argue that the differences between the logics result from LP’s having a formal account of consequence, and LA a material account of consequence. Further, we show that, like many attempts to extend LP with a detachable conditional, Asenjo and Tamburino’s attempt to add a detachable conditional to the logic fails due to Curry’s paradox.

Suppose that you held a view on language in which some predicates are essentially non-classical (i.e., cannot be treated along classical-logic lines). Suppose too that instead of such non-classical predicates yielding ‘gaps’ (where, of some object, the predicate is neither true nor false), you take them to instead deliver the dual of gaps, namely, ‘gluts’ – predicates true and false of objects. Your view might be tied to familiar paradoxical predicates, holding that predicates like ‘is true’, ‘is a member of’, or ‘exemplifies’ are essentially glutty predicates: they cannot be (properly) interpreted in a way that avoids there being objects of which these predicates are both true and false. But you mightn’t think all such predicates are essentially glutty. You might think that some – in fact, the remaining (and majority of) – predicates of the language are essentially classical.

One might think that natural languages are much as above (some essentially glutty predicates, but the remaining predicates essentially classical), or one might merely take it that natural languages could be as such. Either way, the question arises: how should the logic of such a language be understood?

We know that classical logic cannot provide us with a way to formally model such a language. A paraconsistent approach needs to be considered,
where a logic is paraconsistent if and only if arbitrary $\varphi$ and $\neg \varphi$ fail to jointly imply arbitrary $\psi$.

There are well-known paraconsistent logics designed to give a consequence relation for such a language. The logic best known in philosophy for such an application is the logic of paradox (LP), discussed explicitly and widely by Priest [13, 15];\(^1\) this logic continues to gain attention in the philosophy of logic and beyond. A closely related but far less familiar and equally less explored approach is the logic of antinomies (LA) offered by Asenjo and Tamburino [5].\(^2\) At first glance, the account offered by Asenjo and Tamburino seems radically different from what LP delivers. But it isn’t, and the relationship between LP and LA is worth highlighting – our main task in this paper.

In this paper, we make both a philosophical-cum-historical point and also a logical point. Towards the philosophical-cum-historical point, we argue that the differences between LA and LP are not great; they are not rooted in substantive philosophical disagreement, but arise merely from a difference in focus. In particular, LP is the logic for a paraconsistent language when one is concerned with the purely formal notion of consequence, while LA gives the logic when one is concerned with a material notion of consequence. Second, we observe the logical point that the two proposals share the same fundamental weakness in failing to contain a detachable conditional (i.e., a conditional $\rightarrow$ such that $\varphi$ and $\varphi \rightarrow \psi$ jointly imply $\psi$). A notable feature of Asenjo and Tamburino’s proposal is that they attempt the familiar remedy of adding a detachable conditional to the basic boolean framework. In [4], Asenjo comments that this attempt will not work, but does not describe the details of the failure. We make things explicit: we show explicitly that, like many other attempts, the Asenjo–Tamburino approach succumbs to Curry’s paradox, and thus fails to be a viable solution to the problem of detachment.

The discussion is structured as follows. §§1–2 present the target logics in terms of familiar model theory. §3 discusses the main logical differences in

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\(^1\)LP is the gap-free extension of FDE, the logic of tautological entailments; it is the dual of the familiar glut-free extension of FDE called ‘strong Kleene’ or ‘K3’. See Dunn [9, 10], Anderson and Belnap [1], and Anderson, Belnap, and Dunn [2].

\(^2\)For purposes of accommodating glutty theories, the propositional logic LP was first advanced in 1966 by Asenjo [3] under the name calculus of antinomies; it was later advanced, for the same purpose, under the name ‘logic of paradox’ by Priest [13], who also gave the first-order logic under the same name (viz., LP). What we are calling ‘LA’ is the first-order (conditional-free) logic advanced by Asenjo and Tamburino [5], which was intended by them to be a first-order extension of Asenjo’s basic propositional logic. Due to what we call the LA predicate restriction (see page 3), LA isn’t a simple first-order extension of Asenjo’s propositional LP – as will be apparent below (see §3).
terms of difference in philosophical focus. §4 closes by discussing the issue of detachment.

1 The Logic of Antinomies

The logic of antinomies (LA) begins with a standard first-order syntax. The logical vocabulary is $\lor, \neg, \forall$. Constants $c_0, c_1, \ldots$ and variables $x_0, x_1, \ldots$ are the only terms. The set $P$ of predicate symbols is the union of two disjoint sets of standard predicate symbols: $\mathbb{A} = \{A_0, A_1, \ldots\}$ and $\mathbb{B} = \{B_0, B_1, \ldots\}$. (Intuitively, $\mathbb{A}$ contains the essentially classical predicates and $\mathbb{B}$ the essentially non-classical, essentially glutty predicates.) The standard recursive treatment defines the set of sentences.

An LA interpretation $I$ consists of a non-empty domain $D$, a denotation function $d$, and a variable assignment $v$, such that:

- for any constant $c$, $d(c) \in D$,
- for any variable $x$, $v(x) \in D$,
- for any predicate $P$, $d(P) = \langle P^+, P^- \rangle$, where $P^+ \cup P^- = D$.

The only difference from the standard LP treatment appears here, in the form of a restriction that captures the distinction between the essentially glutty and essentially classical predicates:

**LA Predicate Restriction.** For any predicate $P$:

- if $P$ is in $\mathbb{A}$, then the intersection $P^+ \cap P^-$ must be empty;
- if $P$ is in $\mathbb{B}$, then the intersection $P^+ \cap P^-$ must be non-empty.

As above, the $A_i$s are the essentially classical predicates, while the $B_i$s are those which are essentially glutty.\(^4\)

$|\varphi|_v$ is the semantic value of a formula $\varphi$ with respect to a variable assignment $v$, which is defined in the standard recursive fashion. (We leave the relevant interpretation implicit, as it will always be obvious.) For atomics:

$$|Pt|_v = \begin{cases} 0 & \text{if } I(t) \notin P^+ \text{ and } I(t) \in P^- \\ 1 & \text{if } I(t) \in P^+ \text{ and } I(t) \notin P^- \\ \frac{1}{2} & \text{otherwise.} \end{cases}$$

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\(^3\)For simplicity, we focus entirely on unary predicates. Both LA and LP cover predicates of any arity, but focusing only on the unary case suffices for our purposes.

\(^4\)Asenjo and Tamburino’s [5] presentation is rather different; but we present their account in a way that affords clear comparison with LP.
The inductive clauses are as follows:

1. $|\varphi \vee \psi|_v = \max\{|\varphi|_v, |\psi|_v\}$.
2. $|\neg\varphi|_v = 1 - |\varphi|_v$.
3. $|\forall x\varphi|_v = \min\{|\varphi|_{v'} : v' \text{ is an } x\text{-variant of } v\}$.

Conjunction and existential quantification can be defined from these in the normal way.

LA logical consequence is defined as preservation of designated value, where the designated values are 1 and $\frac{1}{2}$. Thus, $\Gamma \vdash_{LA} \varphi$ holds (i.e., $\Gamma$ implies/entails $\varphi$ according to LA) if and only if no LA interpretation designates everything in $\Gamma$ and fails to designate $\varphi$.

## 2 The Logic of Paradox

We obtain the logic LP simply by dropping the LA predicate restriction, but leaving all else the same. Thus, for purposes of ‘semantics’ or model theory of LP, there’s no difference between $A$-predicates and $B$-predicates: they’re all treated the same.

## 3 Contrast: LA and LP

We begin with formal contrast. While both logics are paraconsistent (just let $|\varphi|_v = \frac{1}{2}$ and $|\psi|_v = 0$, for at least some formalæ $\varphi$ and $\psi$), there are some obvious but noteworthy formal differences between the logics LA and LP. LP permits the existence of a maximally paradoxical object – an object of which every predicate is both true and false – whereas LA does not. Indeed, LA – but not LP – validates ‘explosion’ for certain contradictions; for example, for any $A_i$ in $A$ and any $\varphi$,

$$A_i t \land \neg A_i t \vdash_{LA} \varphi.$$  

Similarly, LA validates the parallel instances of detachment (modus ponens):

$$A_i t, A_i t \supset \varphi \vdash_{LA} \varphi$$

where $\varphi \supset \psi$ is defined as usual as $\neg\varphi \vee \psi$. But LP is different: not even a restricted version of detachment is available [8].

As a final and nicely illustrative example, LA validates some existential claims that go beyond those involved in classical logic (e.g., $\exists x(\varphi \lor \neg \varphi)$, etc.),
whereas LP does not. To see this, note that, for any \( B_i \) in \( \mathbb{B} \), the following is a theorem of LA:

\[
\exists x B_i x.
\]

Since any predicate \( B_i \) in \( \mathbb{B} \) must have at least some object in the intersection of its extension and antiextension, it follows that something is in its extension.

### 3.1 Formal versus Material Consequence

How are we to understand the formal differences between LA and LP? We submit that they are naturally understood as arising from different notions of consequence: namely, material and formal consequence. The distinction may not be perfectly precise, but it is familiar enough.\(^5\) Material consequence relies on the ‘matter’ or ‘content’ of claims, while formal consequence abstracts away from such content. Example: there is no possibility in which ‘Max is a cat’ is true but ‘Max is an animal’ is not true; the former entails the latter if we hold the meaning – the matter, the content – of the actual claims fixed. But the given entailment fails if we abstract away from matter (content), and concentrate just on the standard first-order form: \( Cm \) does not entail \( Am \).

The notion of formal consequence delivers conclusions based on logical form alone. Material consequence essentially requires use of the content of the claims or the meaning of things like predicates that appear in them. One way to understand Asenjo and Tamburino’s proposal is that it gives the material consequence relation of a language with essentially glutty predicates, whereas LP gives the formal consequence relation of such a language – abstracting from the ‘matter’ of ‘essential gluttness’ to mere form.

Recall that LA, as defined above, restricts the interpretation of predicates (see page 3). It’s precisely LA’s predicate restriction which delivers the canvassed logical differences between LA and LP. But the restriction is not a purely formal matter: that a predicate is essentially glutty or essentially classical depends on its meaning. As Kripke [11] taught us, predicates do not wear their paradoxicality – their antinomic nature – on their formal sleeves. If we ignore content, and focus just on purely formal features of sentences, LA’s predicate restriction falls away as unmotivated. And that’s precisely what happens in LP, where we abstract away to pure form; the content of predicates doesn’t matter. In other words, the material approach

\(^5\)See [16, Ch. 2] wherein Read provides a defense of material consequence as logical consequence, and also for further references.
to consequence sees logic as constrained by meaning of predicates, whereas the formal approach takes logic to be entirely unconstrained by the meaning of predicates. While this way of putting the distinction is not perfect, it suffices to give what we take to be the general philosophical difference behind the logical differences in LA and LP.

4 Detachment

A salient problem for LP is that there is no detachable (no modus-ponens-satisfying) conditional definable in the logic [8]; and thus, historically, LP has been viewed as unacceptably weak for just that reason. A lesson one might try to draw from the above observations is that LP can be improved by shifting focus to the material notion of consequence. But this is not quite right. Though one fragment of LA differs from LP in that it satisfies detachment, LA is like LP in that detachment doesn’t hold generally: arguments from \( \varphi \) and \( \varphi \rightarrow \psi \) to \( \psi \) have counterexamples.

On this score, Asenjo and Tamburino [5], along with Priest [13, 15], have a solution in mind. The remedy is to add logical resources to the base framework to overcome such non-detachment.\(^6\) But the remedy offered by Asenjo and Tamburino doesn’t work, as we now briefly indicate.

Asenjo and Tamburino define a conditional \( \rightarrow \) that detaches (i.e., \( \varphi \) and \( \varphi \rightarrow \psi \) jointly imply \( \psi \)). The conditional is intended to serve the ultimate purpose of the logic, namely, to accommodate paradoxes in non-trivial theories (e.g., theories of naïve sets or etc), and is defined thus:

\[
|\varphi \rightarrow \psi|_v = \begin{cases} 
0, & \text{if } |\psi|_v = 0 \text{ and } |\varphi|_v \in \{\frac{1}{2}, 1\} \\
\frac{1}{2}, & \text{if } |\psi|_v = \frac{1}{2} \text{ and } |\varphi|_v \in \{\frac{1}{2}, 1\} \\
1, & \text{otherwise.}
\end{cases}
\]

The resulting logic, which we call LA\(\rightarrow\), enjoys a detachable conditional. In particular, defining \( \vdash_{LA\rightarrow} \) as above (no interpretation designates the premise set without designating the conclusion), we have:

\[
\varphi, \varphi \rightarrow \psi \vdash_{LA\rightarrow} \psi.
\]

The trouble, however, comes from Curry’s paradox. Focusing on the set-theoretic version (though the truth-theoretic version is the same), Robert

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\(^6\)Until very recently [7], all LP-based glut theorists focused their efforts on the given task: adding logical resources to the base LP framework to overcome its non-detachment. Whether this is the appropriate response to the non-detachment of LP is something we leave open here.
K. Meyer et al [12] showed that, assuming standard structural rules (which are in place in LP and LA→ and many other logics under discussion), if a conditional detaches and also satisfies ‘absorption’ in the form

\[ \varphi \rightarrow (\varphi \rightarrow \psi) \vdash \varphi \rightarrow \psi \]

then the given conditional is not suitable for underwriting naïve foundational principles. In particular, in the set-theory case, consider the set

\[ c = \{ x : x \in x \rightarrow \bot \} \]

which is supposed to be allowed in the Asenjo and Tamburino (and virtually all other) paraconsistent set theories.\(^7\) By unrestricted comprehension (using the new conditional, which is brought in for just that job), where \( \leftrightarrow \) is defined from \( \rightarrow \) and \( \land \) per usual, we have

\[ c \in c \leftrightarrow (c \in c \rightarrow \bot) \].

But, now, since the Asenjo–Tamburino arrow satisfies the given absorption rule, we quickly get

\[ c \in c \rightarrow \bot \]

which, by unrestricted comprehension, is sufficient for \( c \)'s being in \( c \), and so

\[ c \in c \].

But the Asenjo–Tamburino arrow detaches: we get \( \bot \), utter absurdity.

The upshot is that while LA may well be sufficient for standard first-order connectives, the ‘remedy’ for non-detachment (viz., moving to LA→) is not viable: it leads to absurdity.\(^8\) Other LP-based theorists, notably Priest \([14]\) and subsequently Beall \([6]\), have responded to the non-detachability of LP by invoking ‘intensional’ or ‘worlds’ or otherwise ‘non-value-functional’ approaches to suitable (detachable) conditionals. We leave the fate of these approaches for future debate.\(^9\)

\(^7\) Throughout, \( \bot \) is ‘explosive’ (i.e., implies all sentences).

\(^8\) We note that Asenjo himself noticed this, though he left the above details implicit. We have not belabored the details here, but it is important to have the problem explicitly sketched.

\(^9\) We note, however, that Beall has recently rejected the program of finding detachable conditionals for LP, and instead defends the viability of a fully non-detachable approach \([7]\), but we leave this for other discussion.
5 Closing remarks

Philosophy, over the last decade, has seen increasing interest in paraconsistent approaches to familiar paradox. One of the most popular approaches is also one of the best known: namely, the LP-based approach championed by Priest. Our aim in this paper has been to highlight an important predecessor of LP, namely, the LA-based approach championed first by Asenjo and Asenjo–Tamburino. The Asenjo-Tamburino approach is motivated by the thought that our language has essentially glutty predicates (viz., familiar predicates subject to familiar logico-cum-semantic paradoxes), and that logic needs to account for such essential gluts. The LP-based approach differs only in being more typically ‘purely formal’ – treating all predicates the same. Which, if any, of the two approaches is best for applications to glutty theories is for debate to tell.\(^{10}\)

References


\(^{10}\)We note that Priest’s ultimate rejection of LP in favor of his non-monotonic LPm (elsewhere called ‘MiLP’) reflects a move ‘back’ in the direction of the original Asenjo–Tamburino approach, where one has ‘restricted detachment’ and the like, though the latter logic (viz., LA) is monotonic. We leave further comparison for future debate.
[7] Jc Beall. Free of detachment: logic, rationality, and gluts. Under review; presented at St Andrews, the SEP at The Ohio State University, CUNY Grad Center, the AAP in Sydney, NELLC at Yale, Glasgow, Otago, 2012.


