Leonard, Goodman, and the Development of the *Calculus of Individuals*

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Abstract

This paper investigates the relation of the Calculus of Individuals presented by Henry S. Leonard and Nelson Goodman in their joint paper, and an earlier version of it, the so-called Calculus of Singular Terms, introduced by Leonard in his Ph.D. dissertation thesis *Singular Terms*. The latter calculus is shown to be a proper subsystem of the former. Further, Leonard's projected extension of his system is described, and the definition of an non-extensional part-relation in his system is proposed. The final section discusses to what extent Goodman might have contributed to the formulation of the Calculus of Individuals.

1 The Calculus of Individuals

In 1936, Henry S. Leonard and Nelson Goodman presented a joint paper at the meeting of the Association for Symbolic Logic which was held at the meeting of the Eastern Division of the American Philosophical Association in Cambridge, Massachusetts. Eleven years later, they published an elaborated version of this paper under the title "The Calculus of Individuals and its Uses" [12]. The calculus they introduce in this paper is today usually taken as a basis for the study and use of formal part-whole relations (often called "mereology") in analytic metaphysics, sometimes mediated by Goodman's presentation of the calculus in his *The Structure of Appearance* [7].

Goodman used the Calculus of Individuals in his Ph.D. dissertation thesis *A Study of Qualities* of 1940 [6], which eventually became *The Structure of Appearance*. As in his joint paper with Leonard, he used the calculus as an addition to set theory to solve a problem known as the *difficulty of imperfect community* in Rudolf Carnap's *Aufbau* [1]. Only in *Structure* Goodman abandoned set theory and presented a nominalistic construction that used only the Calculus of Individuals.[1]

The focus of this paper, however, will be an investigation of the system that Leonard presents in his Ph.D. dissertation thesis *Singular Terms* [10] of 1930, which is the first

[1]See [2], 121–139, and [3], §3.2, for a discussion.

occurrence of a system akin to the Calculus of Individuals in this line of development. As is well know, Stanisław Leśniewski developed his mereology, and also coined the term, well before this ([13], [14], [15]), but the work of Leonard and Goodman is independent of Leśniewski's, until W.V. Quine recognised the similarities between the systems.[2] In their joint paper, Leonard and Goodman acknowledge

> The calculus of individuals we shall employ is formally indistinguishable from the general theory of manifolds developed by Leśniewski. ([12], 46)

They go on to justify their project by pointing out that Leśniewski's system is "rather inaccessible, lacks many useful definitions, and is set forth in the language of an unfamiliar logical doctrine and in words rather than symbols" (*ibid.*).

Indeed, Leśniewski's system, Tarski's axiomatisations of it ([20], [21]), the Calculus of Individuals of Leonard and Goodman's paper, and Goodman's version of it that he presents in *Structure*, all turn out to be equivalent.[3] We will now turn to the question, how Leonard's Calculus of Singular Terms fits in.

2 Singular Terms

In his Ph.D. dissertation thesis *Singular Terms* [10] of 1930, Leonard introduced the Calculus of Singular Terms which is an early version of the Calculus of Individuals. Alfred Whitehead supervised Leonard on his Ph.D. project which was conceived of as an extension to Whitehead and Russell's *Principia Mathematica* [22]. The formal sections of his thesis are slotted into those of *Principia* and bear the numbers *16 to *18, and thus sit in the space between the introduction of definite descriptions, *14, and the theory of classes, *20 and following.

Leonard's aim was to bring clarity into the confusion that surrounded the distinction of universal and particular and the conflation of this distinction with that of concreta and abstracta:[4]

> The distinction of particular and universal has suffered from an ambiguity. It has covered (1) the distinction between entities with and entities without spatio-temporality, and (2) the distinction between entities which are complex in the sense of a realistic interpretation and those which are simple. On the last basis, the distinction is relative; on the first, arbitrary, and what is more, the two are not coextensive. ([10], abstract, 4)

He aims to give a precise and formalised treatment the distinction between universal and particular, the beginning of which is the Calculus of Singular Terms, since he deems the apparatus of the theory of classes developed in *Principia Mathematica* unfit to this purpose.

> The calculus of classes has, we said, offered us no treatment of the traditional abstract and universal terms, "redness," "coloredness," "weight," and the

[2] [16], 122; see the quotation in §4 below.

[3] Given some straightforward assumptions; for details see Lothar Ridder's *Mereologie* [17]. See also Rolf Eberle's [4] and [5].

[4] Since Leonard's thesis is unfortunately unpublished, I will quote extensively from it, in order to make the relevant passages available.

like, but only of what have been called in traditional logic concrete general terms, "red things," "colored things," "stones," and the like. And yet we do use such terms, and not only as predicates but as subjects of predication as well. We say, for example, "Scarlet is red," "Scarlet is a color." It is not enough that we can always present a materially equivalent proposition, such as "Scarlet things are red things," or "The class of scarlet things is a member of the class of classes, 'color.'" Ideally our symbolic logic should offer us a calculus which analyses immediately the structure of these propositions that employ abstract terms. And until it does that, our symbolic logic is incomplete. For if it cannot do that, it cannot symbolise the equivalence of these two propositions, and the claim that they are equivalent is a claim made outside the symbolic system, made by the interpretor rather than by the system builder.

> The object of the present paper is to seek a solution to this problem. In general, our solution consists in holding that the calculus of classes is the calculus of general terms, that abstract terms are rightfully excluded from the type called "general," but that they belong to the type called "singular." Thus, instead of omitting them entirely, we add a new chapter to symbolic logic, and it is this addition which characterizes our whole position. ([10], 6–8.)

Leonard thus proceeds to develop the Calculus of Singular Terms as an addition to the system of *Principia Mathematica* and its calculus of classes.

> The calculus of classes achieved success through its disregard of the traditional abstract terms. The object of this thesis is to show that this disregard was justified, because abstract terms are singular and not general, that on the other hand, symbolic logic should and can provide a treatment of these terms in a calculus of singular terms and that with this development, it is possible to see that symbolic logic has not thrown over intension, but only its traditional association with abstract terms. ([10], abstract, 1)

2.1 The Calculus of Singular Terms

The formal system of the Calculus of Singular Terms is based on a term-forming operation in which we nowadays might recognise the binary mereological sum. Leonard prefaced his axiom system with the remark:

> For the development of this chapter, we require one primitive idea, which we introduce as a descriptive function and denote by "$x + y$." By this we denote a symbol of the form "$(\imath x)(\phi x)$," where the specific determinant function is not indicated. [...] By "$x + y$" we mean to describe that individual which arises from the most general togetherness of any two other individuals. ([10], 187)

In other words, '+' is taken as primitive and treated as a definite description. Next, the defined notions are introduced ([10], 190) which are the relations '$<$' for 'is part of', '**o**' for 'overlaps', and two further definite descriptions, namely, concatenation of terms for 'product', and '$-$' for 'difference'.[5]

Definitions

*16.01 $x < y \ =_{df} \ x + y = y$
*16.02 $x \, \mathbf{o} \, y \ =_{df} \ \exists z (z < x \wedge z < y)$
*16.03 $xy \ =_{df} \ (\imath z)(z < x \wedge z < y \wedge \forall t[(t < x \wedge t < y) \supset t < z])$
*16.04 $x - y \ =_{df} \ (\imath z)(\neg z \, \mathbf{o} \, y \wedge (x \, \mathbf{o} \, y \supset z + xy = x) \wedge (\neg x \, \mathbf{o} \, y \supset z = x))$

Next the postulates, or axioms, that govern '+' are listed ([10], 191):

Postulates

*16.1 $x + y = y + x$ (PM *22.57)
*16.12 $\mathrm{E}!x + x \supset x + x = x$ (PM *22.56)
*16.14 $[\mathrm{E}!x + y \wedge \mathrm{E}!y + z \wedge \mathrm{E}!(x + y) + z \wedge \mathrm{E}!x + (y + z)] \supset (x + y) + z = x + (y + z)$ (PM *22.7)
*16.16 $x \, \mathbf{o} \, y \supset \mathrm{E}!xy$ (PM *22.36)
*16.17 $x \, \mathbf{o} \, (y + z) \supset (x \, \mathbf{o} \, y \vee x \, \mathbf{o} \, z)$
*16.18 $x < (y + z) \supset (x < y \vee x < z \vee x < xy + xz)$

The numbers in the right hand column (PM *22.57, etc.) denote the corresponding propositions of the calculus of classes of *Principia Mathematica.* Leonard gives these references throughout his presentation.

Note that 'E!' is the existence predicate of *Principia* [22], which is defined as

PM *14.02 $\mathrm{E}!(\imath x)(\phi x) \ =_{df} \ \exists b \forall x (\phi x \equiv x = b)$

Sum, product and difference are definite descriptions, as noted above. As such they are guaranteed to refer to at most one object, but it is not thereby guaranteed that they pick out an object at all. If we want to read '+', etc., as functions, we thus have to take them to be *partial* functions. Indeed, the product must not be a total function; the existence of the product of some objects x and y is only guaranteed, if x and y overlap (*16.16).

The first theorem that Leonard proves is that, in fact, any two objects have a sum:

*16.2 $\mathrm{E}!x + y$

As Leonard explains ([10], 192), *16.2 follows immediately from *16.1 by the PM proposition

PM *14.21 $\psi(\imath x)(\phi x) \supset \mathrm{E}!(\imath x)(\phi x)$

[5]Leonard gives the formalism in the notation of *Principia Mathematica* [22]; for ease of legibility, the notation is modernised in the exposition below. Note that '$(\imath x)$' is the definite description operator.

The first postulate, *16.1, thus packs two into one: the commutativity of the sum and the existence of arbitrary sums. Leonard notes that *16.1 could have been split into two postulates: one in the form of theorem *16.2, and the other in a conditionalised form analogous to *16.12 and *16.14. With the help of *16.2, the antecedents of *16.12 and *16.14 can now be discharged.

Leonard proceeds to prove a large number of theorems which are familiar to us today as theorems of classical mereology. Here is a (comparatively) small selection:

Some theorems

*16.21	$\exists z\,(x+y)=z$
*16.213	$\mathrm{E}!(x+y)+z$
*16.214	$(x+y)+z=x+(y+z)$
*16.221	$(x+y)=z \supset x<z$
*16.25	$x<x$
*16.26	$x<(x+y) \wedge y<(x+y)$
*16.271	$x=y \supset z<x \equiv z<y$
*16.274	$x=y \equiv (x<y \wedge y<x)$
*16.28	$(x<z \wedge y<z) \equiv x+y<z$
*16.281	$(x<y \wedge y<z) \supset x<z$
*16.3	$x\,\mathbf{o}\,y \equiv \exists z(z<x \wedge z<y)$
*16.31	$(x<y \wedge x<z) \supset y\,\mathbf{o}\,z$
*16.32	$x\,\mathbf{o}\,y \equiv y\,\mathbf{o}\,x$
*16.321	$x<y \supset (x\,\mathbf{o}\,y\ \wedge\ y\,\mathbf{o}\,x)$
*16.323	$x\,\mathbf{o}\,x$

2.2 Universals and the *crossing* relation

Next, Leonard sets out to incorporate universals into the system. He introduces the crossing relation to this end.

> By "x k y" (read "x crosses y"), we denote a specific propositional form, corresponding in meaning to what "y is x" would often mean. As we shall point out in the next two chapters, this symbol is amenable to a variety of interpretations and in this variety shows that logic itself does not dictate a philosophic world-view. [...] [On a realist] interpretation, "x k y" means that "x" stands to "y" in the relation of an ingressing quality to the object it constitutes, or partially constitutes, that "x" qualifies, or is a quality of, "y." Thus, for example, we should say that whiteness crosses this sheet of paper." ([10], 214–215.)

Using 'crossing', other notions become definable, for example, 'uniform crossing' ('$\mathrm{k_u}$'), 'portion' ('$'$'), and 'occuring together' of qualities ('w'):

*17.01	$x\ \mathrm{k_u}\ y\ =_{df}\ \forall z(z<y\ \supset\ x\ \mathrm{k}\ z)$
*17.201	$x\,'y\ =_{df}\ (\iota z)(x\ \mathrm{k_u}\ z \wedge z<y \wedge \forall t[(x\ \mathrm{k_u}\ t \wedge t<y) \supset t<z])$
*17.301	$x\ \mathrm{w}\ y\ =_{df}\ \neg x\,\mathbf{o}\,y \wedge \exists z(x\ \mathrm{k}\ z\ \wedge\ y\ \mathrm{k}\ z\ \wedge\ x\,'z\,\mathbf{o}\,y\,'z)$

Leonard does not present an axiom system of the crossing relation and the related notions. He merely lists some desiderata for such an axiomatisation and remarks:

> It is our ambition to work out something more definite for this science in the next few years. We offer the following as a suggestion of the direction in which we should seek a solution. ([10], 214)

Alas, Leonard never returned to this project in print. Amongst the desiderata he lists are ([10], 217):

*17.1	$(x \text{ k}_\text{u}\, y \wedge y \text{ k}_\text{u}\, z) \supset x \text{ k}_\text{u}\, z$
*17.11	$(x \text{ k}_\text{u}\, y \wedge y \text{ k } z) \supset x \text{ k } z$
*17.12	$(x \text{ k}_\text{u}\, y \wedge z \text{ k}_\text{u}\, t) \supset (x + z) \text{ k}_\text{u}\, (y + t)$
*17.13	$(x \text{ k } y \wedge y < z) \supset x \text{ k } z$
*17.14	$(x \text{ k } y \wedge x < z) \supset z \text{ k } y$
*17.142	$x \text{ k}_\text{u}\, y \supset x \text{ k } y$

Leonard remarks that "which of these must be taken as primitive propositions [i.e., postulates, or axioms], we do not as yet know" ([10], 217). He also observes, however, that using *17.13 and *17.14, together with *16.2, the existence of arbitrary sums, it is possible to derive the undesirable consequence

*17.141 $x \text{ k } y \supset (x + y) \text{ k } (x + y)$

which says that any entity is a quality of itself. Leonard discusses two responses, both of which he finds unsatisfactory. The first is to bite the bullet on the result; the second, to restrict summation to individuals that are in the same category.

> The difficulty is that we do not know what a category is. It would seem to us that logic cannot know more about categories than what can be formulated in such abstract rules as these which we are giving. As a result, I do not at present see just how the limitations for the existence of sums can be formulated. ([10], 218)

A further desideratum has even more severe problems:

*17.15 $x \text{ k } y \supset \neg x \mathbf{o}\, y$

"is inconsistent with those which we have already given, for '$x + y \;\mathbf{o}\; x + y$' is true by *16.33, and is false by *17.141 [and *17].15. But by denying *16.2, we could, by this theorem, deny *17.141" ([10], 218).

Being a portion of something is not to be identified with being a part of this object. Rather, x is a part of y if and only if any quality of x is a quality of y:

*17.16 $x < y \equiv \forall z(z \text{ k } x \supset z \text{ k } y)$

which "says that the whole has all the qualities of the part and that if one thing had all the qualities of another, then it must be the whole of which that other is a part" ([10], 219). Desiderata for 'portion' ('$'$') and 'occuring together' ('w') follow, as does a "more or less speculative" definition of "that unit which two units occurring together constitute in virtue of their occurence together", '$\underline{\mathrm{w}}$', (think of product and overlap) ([10], 221):

$$\text{*17.401} \quad x \ \underline{\mathrm{w}}\ y \ =_{df} \ (\iota z)(\forall t[t \ \mathrm{k} \ z \supset (t < x \vee x < t \vee t \ \mathrm{k} \ x \vee x \ \mathrm{k} \ t \vee t < y \vee t < y \vee t \ \mathrm{k} \ y \vee y \ \mathrm{k} \ t)])$$

*18 is developed in an even more rudimentary way. It gestures at defining 'complex' ('ex', as in 'complex') and makes use of the calculus of classes (and should thus rather appear after *24 in the system of *Principia Mathematica*). To define 'ex', the auxiliary definition which extends the definition of '$\underline{\mathrm{w}}$' given above to the case of classes, is introduced:

$$\text{*18.01} \quad x \ \underline{\mathrm{w}}\ \alpha \ =_{df} \ (\iota z)(\forall y[y \in \alpha \supset \forall t((x \ \mathrm{w} \ y \wedge (x \ \underline{\mathrm{w}}\ y) \ \mathrm{k} \ z \wedge (x \ \underline{\mathrm{w}}\ y) \ \mathrm{k} \ t) \supset z < t)])$$
$$\text{*18.02} \quad x \ \mathrm{ex} \ =_{df} \ x \ \underline{\mathrm{w}}\ \hat{y}(y \ \mathrm{w} \ x)$$

Note that 'α' is a class variable and that '$\hat{x}$' is the *Principia* notation for class abstraction.

The complexity of the planned extension of Leonard's project should be sufficiently clear at this point.

2.3 Pluralism in *Singular Terms*

As we could already see in the first quote given in the section, Leonard's philosophy is pluralistic in a way that will be characteristic for Goodman's philosophy too (cf. [3] and chapter 8 of [2]). In the final chapter of *Singular Terms*, this feature of his views becomes even more pronounced:

> All units are equally real, but in order to introduce system in our view of the world, we must take certain ones as basic and describe others in terms of these. In the suggestions which we have just outlined, we have taken quality units as basic, where under "quality units" we include units of space and time. In terms of these, we describe units of other types. On this view quality units are real parts of our world, the basic units in a world view. ([10], 238)

Specifically, this pluralism pertains to the nominalism-realism debate:

> Realism says that the quality crosses the object, nominalism that the object crosses the quality.
>
> Let us consider for a moment some uniform object, say a white billiard ball. It is uniformly white, uniform in density and in hardness. There are, of course, a multitude of other qualities but let us restrict ourselves to these three. We may say that the billiard ball is these qualities occurring with each other and with a certain space-time unit. This would be the realistic view [...]. On the nominalistic view we should reverse the account. We should say that whiteness is the complex of all white objects occurring together. On the realistic view objects do not occur together, but form a system of units which have relations of part and whole, discreteness, and the like. One the nominalistic view, objects do occur together, and in their occurrence together constitute quality units, which are thereby derivative and relatively complex. ([10], 238–239)

Realism and nominalism, thus, appear merely as different ways to interpret the formal system that characterises the relation between universals and particulars ([10], 241 & 247). But Leonard is prepared to adopt an even more far-reaching position, clearly foreshadowing Goodman's pluralistic irrealism of *Ways of Worldmaking* [9]. In section D of the penultimate chapter of *Singular Terms* (242–247) Leonard describes how choosing different units as basis of the system he presents can serve different purposes. He mentions different special sciences that might require different such unitations, and also discusses an application to the arts—a further analogy to Goodman. In sum, he asks,

> why should every function of life be suited by the same mode of unitation and why should even every branch of knowledge and culture find the same mode most satisfactory? Why, finally, should philosophy be concerned to develop a new and distinctive unitation? Is it not possible that the end of philosophy would be the careful correlation of those systems of unitation which other disciplines and other cultures spin out or exemplify? ([10], 245)

This is, however, not the place to investigate Leonard's pluralism any further. Instead, let us venture into the formal comparison of the Calculus of Singular Terms with the Calculus of Individuals.

3 The Calculus of Individuals *vs* the Calculus of Singular Terms

For the formal comparison of the two calculi, the Calculus of Individuals too will briefly be described. Leonard and Goodman here take discreetness, '$\rceil$', as primitive relation ([12], 46) and define parthood ('$<$'), overlap ('$\mathbf{o}$'), fusion ('Fu') and sum ('$+$'), and nucleus ('Nu') and product (by concatenation of terms). Note that fusion and nucleus are generalised notions of binary sum and product, respectively.

Definitions

I.01 $x < y \; =_{df} \; \forall z(z \rceil y \supset z \rceil x)$
I.02 $x \,\mathbf{o}\, y \; =_{df} \; \exists z(z < x \wedge z < y)$
I.03 $x \text{ Fu } \alpha \; =_{df} \; \forall z(z \rceil x \equiv \forall y(y \in \alpha \supset z \rceil y))$
I.04 $x \text{ Nu } \alpha \; =_{df} \; \forall z(z < x \equiv \forall y(y \in \alpha \supset z < y))$
I.06 $x + y \; =_{df} \; \text{Fu‘}(\{x\} \cup \{y\})$
I.07 $xy \; =_{df} \; \text{Nu‘}(\{x\} \cup \{y\})$

'Fu‘ ' and 'Nu‘ ' are so-called *descriptive functions*, introduced in *Principia Mathematica*:

PM *30.01 $R\text{‘}y \; =_{df} \; (\imath x)(xRy)$

Thus, since 'x Fu α' can be read as 'x fuses the members of α', 'Fu‘ α' should be understood as 'the fusion of the members of α'.

I.03–I.07 mention classes: a class variable, 'α', occurs in I.03 and I.04, on which I.06 and I.07 build. Leonard and Goodman suggest that instead of the definitions mentioning classes the following could be used ([12], 48):

I.06′ $x + y \ =_{df} \ (\iota z)(\forall w[w \mathrel{\rceil\!\!\lfloor} z \ \equiv \ (w \mathrel{\rceil\!\!\lfloor} x \wedge w \mathrel{\rceil\!\!\lfloor} y)])$
I.07′ $xy \ =_{df} \ (\iota z)(\forall w[w < z \ \equiv \ (w < x \wedge w < y)])$

The three axioms they propose are ([12], 48–49):

Postulates

I.1 $\exists x\, x \in \alpha \supset \exists y\, y \text{ Fu } \alpha$
I.12 $(x < y \wedge y < x) \supset x = y$
I.13 $x \,\mathbf{o}\, y \ \equiv \ \neg x \mathrel{\rceil\!\!\lfloor} y$

Some Theorems

I.3 $(x < y \wedge y < z) \supset x < z$
I.31 $x < x$
I.331 $x \,\mathbf{o}\, y = y \,\mathbf{o}\, x$
I.332 $x < y \supset x \,\mathbf{o}\, y$
I.333 $x \,\mathbf{o}\, x$
I.53 $\exists x\, x \in \alpha \equiv \text{E!Fu‘}\alpha$
I.6 $\text{E!}x + y$
I.62 $x + y = y + x$
I.66 $(x + y) + z = x + (y + z)$

It is relatively straightforward to see that the Calculus of Individuals satisfies all definitions of the Calculus of Singular Terms and also proves all axioms of the latter. The Calculus of Singular Terms is thus a subsystem of the Calculus of Individuals. The converse is much harder to establish: in fact, it is impossible. Neither I.01 (or rather, the right-to-left direction of the definitional equivalence) nor I.06′ can be derived in Leonard's system. The Calculus of Singular Terms is hence a *proper* subsystem of the Calculus of Individuals. (See Appendix A for a proof sketch.)

The definitions of 'overlap' and 'product' fall short of the characterisation that these notions are given in the Calculus of Individuals, which we nowadays associate with classical extensional mereology (which, essentially, is a Boolean algebra with the zero-element removed[6]).

3.1 Non-extensional parthood

The "shortcoming" in the definition of 'overlap' in the Calculus of Singular Terms might turn out to be a blessing in disguise, however. Using 'overlap', it is possible to define in this system a "non-extensional"[7] part-relation in the following way (compare I.01):

$\text{D}_{\prec}$ $x \prec y \ =_{df} \ \forall z(z \,\mathbf{o}\, x \supset z \,\mathbf{o}\, y)$

Call the resulting system $\text{ST}^{\prec}$. This non-extensional part-relation, '$\prec$', might be of interest in contemporary discussions in analytic metaphysics: think of the problem of the statue and the clay. A statue and the lump of clay it consists of arguably share —

[6]See Eberle [5], 36, and Simons [19], 25; see Ridder [17], chapter 3, for a detailed investigation.
[7]Following the terminology suggested in Simons' [19].

at least for some given time, t, say — all their parts. Nevertheless, one might want to resist identifying the statue with the clay. The thought behind this is that the statue will survive the loss of a bit of matter, while the lump of clay will not. At the given time t the statue and the clay will coincide, however, in the sense that they completely overlap each other. But in the Calculus of Individuals and other classical extensional mereologies complete overlap means identity:

$$\forall z(x \,\mathbf{o}\, z \wedge y \,\mathbf{o}\, z) \supset x = y$$

Conveniently, in the Calculus of Singular Terms this does not hold, and $\prec$ inherits this non-extensionality of $\mathbf{o}$. Thus, while mutual extensional parthood means identity:

$$(x < y \wedge y < x) \supset x = y$$

this is not true of the non-extensional part-relation:

$$\nvdash (x \prec y \wedge y \prec x) \supset x = y$$

So, while the statue and the lump of clay can share all $\prec$-parts and be $\prec$-part of each other, this does not entail that they are identical.

This is not the place to investigate the philosophical potential of the here introduced $\prec$-relation. Note, however, before we move on, that $\prec$ appears to have all right to be called a part-relation, since it has many of the characteristic properties. In particular, the following theorems hold (see Appendix B for hints on the proofs):

Some theorems of ST$^{\prec}$

ST$^{\prec}$.1	$x \prec x$	(compare *16.25)
ST$^{\prec}$.2	$(x \prec y \wedge y \prec z) \supset x \prec z$	(compare *16.281)
ST$^{\prec}$.3	$x < y \supset x \prec y$	
ST$^{\prec}$.4	$(x \prec y \wedge x \,\mathbf{o}\, z) \supset y \,\mathbf{o}\, z$	
ST$^{\prec}$.5	$(x \prec y \wedge z \prec x) \supset y \,\mathbf{o}\, z$	
ST$^{\prec}$.6	$(x \prec y \wedge x \prec z) \supset y \,\mathbf{o}\, z$	(compare *16.31)
ST$^{\prec}$.7	$x \prec y \supset (x \,\mathbf{o}\, y \wedge y \,\mathbf{o}\, x)$	(compare *16.321)
ST$^{\prec}$.8	$x \prec (x + y)$	(compare *16.26)
ST$^{\prec}$.9	$(x \prec z \wedge y \prec z) \supset (x + y) \prec z$	(compare *16.28, l.t.r.)

Extensionality fails for $\prec$ however. Accordingly, $\prec$ is a proper subrelation of $<$.

$\nvdash (x \prec y \wedge y \prec x) \supset x = y$	($\prec$ is non-extensional)
$\nvdash x \prec y \supset x < y$	($\prec$ and $<$ do not coindcide)

Note further:

$\nvdash x \prec y \supset y \prec x$	($\prec$ is not symmetric)
$\nvdash x \prec y \supset \neg y \prec x$	($\prec$ is not antisymmetric)

4 Goodman's input

Despite not being discussed in print, it has been subject of speculation for a long time, what exactly Goodman's contribution to the Calculus of Individuals was. It seems clear that the third section of Leonard and Goodman's paper [12], which deals with the application of the calculus to Carnap's *Aufbau* and the solution of its difficulty of imperfect community, was largely Goodman's work. It is an open question, however, to what extent Goodman contributed to the construction of the calculus itself and how much of it was developed by Leonard. Some light has been shed on the question here by outlining how much of the system was already contained in *Singular Terms*.

The problem is complicated by the fact that Leonard co-operated with Goodman already before he finished his Ph.D. dissertation thesis. Leonard thanks Goodman in the preface of *Singular Terms*, and states that the two "discussed together nearly every point developed in this thesis" ([10], v). Indeed, in *The Structure of Appearance*, Goodman remarks in a footnote:

> The calculus to be outlined here was developed by Henry S. Leonard and Nelson Goodman. It was first presented in Leonard's doctoral thesis *Singular Terms* [...]. ([7], 33, fn. 8)

The reference to Leonard's *Singular Terms* is missing from Goodman's *A Study of Qualities* [6]. In *Problems and Projects* Goodman vaguely concedes:

> The first thought of developing some of the basic ideas by the use of the (then rather new) techniques of symbolic logic came from my fellow student Henry Leonard, and our collaboration led to "The Calculus of Individuals and its Uses". ([8], 149)

Recently a document surfaced that might be of help. In 1966, Leonard started preparing a collection of his papers for publication; a project that was never finished since Leonard sadly died of a heart attack on July 11, 1967, aged 61, on a vacation in Frankfurt, Germany. The collection was meant not only to contain his joint paper with Goodman, but also introductory comments on each each paper. The comment on "The Calculus of Individuals and its Uses" was written, and is now due to be published (see [11]). In this comment, Leonard writes:

> Although Goodman and I published "The Calculus of Individuals" only in 1940, such a calculus had for a long time been occupying our attention, both independently and collaboratively. Concern with a part-whole relation between individuals was a major one in Goodman's Honors Thesis, submitted to the Harvard Department of Philosophy when he was a senior in 1928. A formal development of the calculus constituted Chapter IV of my doctoral dissertation, submitted to the Harvard department in December, 1930. In the fall of that year, as I was writing the thesis, Goodman and I met together many times for exchanges of ideas. [...]
>
> The earliest version of the calculus, in my dissertation in 1930, differed from the later versions in certain significant respects. It was presented as an interpolation in Whitehead and Russell's *Principia Mathematica* between *14 and *20. Hence, it did not include such general notions as those of the

> fusion and the nucleus of a class (I.03 and I.04, below). Instead of taking the relational expression '$x \mathrel{\rceil\!\lfloor} y$' (i.e., '$x$ is discrete from y') as primitive, it took the operation '$x + y$' as primitive. It rested on more postulates than did the later versions. (In fact, the 1936 version still had five postulates.) [...]
>
> If responsibilities can be divided in a collaborative enterprise, I believe that it may be fairly stated that the major responsibility for the formal calculus (Part II, below) was mine, while the major responsibility for discussions of applications (Part III) lay with Goodman.

Goodman's Honors Thesis unfortunately appears to be lost; at least, it is not contained in any of the known collections of his *Nachlass*.[8]

Despite Leonard's comments, it is still not entirely clear what impact Goodman had on the Calculus of Individuals. It is also unknown what exactly Quine's role was. Quine writes in his autobiography that his intervention took place in 1935 when Leonard, Goodman and Quine had a stop-over for the night on their way back to Cambridge from the famous trip to Baltimore they made with Carnap:

> Goodman, Leonard, and I drove back to Cambridge, stopping in New York at the Lafayette, a French hotel off Washington Square. On the way they told of a project of theirs. They broached it diffidently, for I had seemed unsympathetic when Henry spoke of it on an earlier occasion. I became interested as I heard more, and I was able to help them on a technical problem. We talked in our hotel room until four in the morning. They were concerned with constructing a systematic theory of sense qualities, and their effort had much in common with Carnap's *Logischer Aufbau der Welt*. As an auxiliary they had developed a logic of part–whole relation, which I recognized as Leśniewskis so called mereology. ([16], 122)

What this technical problem was appears to be lost to history.

The fact that the Calculus of Singular Terms is a proper subsystem of the Calculus of Individuals, and is also formulated without the use of classes in contrast to the latter, opens up the possibility of a significant input of Goodman's. But given the lack of further evidence, this must remain speculation.[9]

Appendix A

To prove that the Calculus of Singular Terms (ST) is a proper subsystem of the Calculus of Individuals (CI), it suffices to show that the definitions and axioms of ST can be derived in CI, but that the converse does not hold.

We leave out of consideration the notions defined in either system that do not interact with the axioms (such as the difference in ST, or the universe, proper part, and complement in CI). ST does not contain the discreteness relation, '$\rceil\!\lfloor$', which, for the purpose at hand, is taken to be defined thus (compare I.13):

[8] Part of Goodman's *Nachlass* is held by the Harvard University Archives, part by his literary executor, Catherine Elgin.

[9] I would like to thank Ralf Bader, Daniel Cohnitz, Philip Ebert and Karl-Georg Niebergall for helpful comments on earlier versions of this paper.

$x \mathrel{\rceil\!\lfloor} y \; =_{df} \; \neg x \,\mathbf{o}\, y$

Further, only the class-free fragment of CI is considered. I.06′ and I.07′ thus replace I.03–I.07; an axiom that asserts the existence of arbitrary binary sums is used in lieu of the fusion axion, I.1, which contains a variable ranging over classes.

I.1′ $\exists z \, x + y = z$

Obviously, I.1 implies I.1′. The tables below show what postulates and definitions are used in CI to derive the axioms and definitions of ST, and *vice versa*.

ST definitions	derived in CI using
*16.01	I.01, I.06′
*16.02	= I.02
*16.03	I.01, I.02, I.06′, I.07′, I.1′, I.12, I.13

ST postulates	derived in CI using
*16.1	(= I.62) I.01, I.02, I.06′, I.1′, I.12, I.13
*16.12	I.06′, I.1′
*16.14	I.01, I.02, I.06′, I.07′, I.12, I.13
*16.16	I.01, I.02, I.06′, I.07′, I.12, I.13
*16.17	I.06′, I.1′, I.13
*16.18	I.01, I.02, I.06′, I.07′, I.12, I.13

Thus, ST is a subsystem of CI. The converse, however, does not hold:

CI definitions	derived in ST using
I.01 l.t.r.	*16.01, *16.02, *16.1, *16.14
I.01 r.t.l.	*fails*
I.02	= *16.02
I.06′	*fails*
I.07′	*16.01, *16.02, *16.03, *16.1, *16.12, *16.14, *16.16

CI postulates	derived in ST using
I.1′	(= *16.21) *16.1
I.12	(cf. *16.274) *16.01, *16.1
I.13	n/a

Appendix B

The system $\mathrm{ST}^{\prec}$ introduced in this paper extends the language of ST by the binary relation symbol, '$\prec$', for non-extensional parthood, and the axiom system by the definition:

$$\mathrm{D}_{\prec} \qquad x \prec y \ =_{df} \ \forall z(z \,\mathbf{o}\, x \supset z \,\mathbf{o}\, y)$$

The following table lists some theorems of $\mathrm{ST}^{\prec}$ and what is required for their proofs:

$\mathrm{ST}^{\prec}$ theorems	derived using $\mathrm{D}_{\prec}$ and:
$\mathrm{ST}^{\prec}.1$	($\mathrm{D}_{\prec}$ suffices)
$\mathrm{ST}^{\prec}.2$	($\mathrm{D}_{\prec}$ suffices)
$\mathrm{ST}^{\prec}.3$	*16.01, *16.02, *16.1, *16.14
$\mathrm{ST}^{\prec}.4$	*16.02
$\mathrm{ST}^{\prec}.5$	*16.01, *16.02, *16.1, *16.12
$\mathrm{ST}^{\prec}.6$	*16.01, *16.02, *16.1, *16.12
$\mathrm{ST}^{\prec}.7$	*16.01, *16.02, *16.1, *16.12
$\mathrm{ST}^{\prec}.8$	*16.01, *16.02, *16.1, *16.14
$\mathrm{ST}^{\prec}.9$	*16.1, *16.17

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