On a Mean Field Haken-Kelso-Bunz Model and a Free Energy Approach to Relaxation Processes

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Nonequilibrium phase transitions in task-related populations of neurons are discussed using a mean field model that involves the Haken-Kelso-Bunz model for bistable motor control systems and the Kuramoto coupling term for coupled phase oscillators. The relation of this model to a general free energy approach to stochastic processes in many body systems is examined.

Key words: Haken-Kelso-Bunz model, Kuramoto mean field coupling, Fokker-Planck equations

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1 Introduction

Many complex systems exhibit collective properties that do not exist on the level of their subsystems. These collective properties typically arise due to self-organization [1, 2, 3]. This article deals with an important class of complex systems. It is concerned with many body systems subjected to noise. More precisely, we consider systems that consist of many interacting subsystems with subsystems of the same kind. The focus will be on isodiffusive processes. For a system that is in the stationary case in thermal equilibrium with its environment this means that the temperature is constant (i.e., we deal with isothermal processes). For a system that operates far away from a state of thermodynamic equilibrium but can exhibit stationary behavior this means that the strength of the fluctuations to which the system is subjected is constant. Let us now distinguish between three kinds of many body systems: systems exhibiting collective properties, systems with probability-dependent diffusion coefficients, and quantum mechanical systems.

Examples of many body systems exhibiting collective properties are: ferromagnetic materials, Van der Waals gases, nematic liquid crystals, populations of insects, populations of neurons, human societies, and so one. Irrespective of their differences, they exhibit under certain conditions collective properties. In ferromagnetic materials spontaneous magnetization can be observed due to spin-spin interactions [4, 5]. Van der Waals gases are characterized by an internal pressure which is absent in ideal gases and is produced by the weak Van der Waals forces between gas atoms [6]. Nematic liquid crystals have a nematic phase with orientational order which is caused by the mutual interactions between the crystal particles [7]. Populations of fireflies are known to synchronize their flashes [8, 9]. Large numbers of neurons synchronize their rhythmic activity in response to the presentation of visual stimuli [10, 11, 12]. There is evidence that this synchronization arises from intercortical and cortico-cortical connections [13]. Moreover, audiences exhibit mass-behavior in terms of synchronized applause [14]. In many cases collective properties emerge when interaction forces dominate fluctuation forces. Whereas interaction forces tend to establish order, noise tends to destroy it. We can express this observation in terms of the free energy $F$ given by

$$F = U - Q S_{\text{BGS}},$$

where $U$, $S_{\text{BGS}}$, and $Q$ are the internal energy, the Boltzmann-Gibbs-Shannon entropy $S_{\text{BGS}}[u] =$
force and can be described by the so-called nonlinear diffusion of populations [22]. In these cases, diffusion systems [20], engineering problems [21], and dispersion [18], diffusion in porous media [19], filtration discussed, for example, in the context of surface diffusion dependent diffusion coefficients have been discussed. For a material property the diffusion coefficient. Denomination depends on the subsystem density depends on the diffusion coefficient but the diffusion coefficient depends on the subsystem density as well. We are inclined to say, that in this case subsystems interact with each other in a indirect fashion. This indirect interaction involves as a material property the diffusion coefficient. Density dependent diffusion coefficients have been discussed, for example, in the context of surface diffusion [18], diffusion in porous media [19], filtration systems [20], engineering problems [21], and dispersal of populations [22]. In these cases, diffusion can be described by the so-called nonlinear diffusion equation [23]. Recently, it has been shown that the nonlinear diffusion is a special case of a general nonlinear diffusion equation [24, 25] that involves the free energy (1) with $S$ given by the generalized entropy $S_q$ proposed by Tsallis [26]. The generalized entropy $S_q$ depends on a single parameter $q$ and recovers the Boltzmann-Gibbs-Shannon entropy $S_{BGS}$ for $q \to 1$. Therefore, we may put Eq. (1) like

$$F = U - QS_q.$$ (3)

So far, the focus was on classical interactions: direct subsystem-subsystem interactions leading to Eq. (2) and indirect interactions related to Eq. (3). In order to take quantum mechanical interactions into account, we may follow Uhling and Uhlenbeck [27, 28] and Kaniadakis [29]. Using a modified Boltzmann equation for fermions and bosons as a departure point, one can derive nonlinear evolution equations for probability densities for many body systems satisfying a statistics different from the Boltzmann-Gibbs-Shannon statistics such as the Fermi-Dirac and the Bose-Einstein statistics. It does not come as a surprise that these evolution equations are related to free energies that embody generalized entropies such as the entropies $S_{BE}$ and $S_{FD}$ [30] for bosons and fermions, respectively.

In order to encompass systems characterized by all kinds of entropies such as $S_{BGS}$, $S_q$, $S_{BE}$ and $S_{FD}$, we modify Eq. (3) again and write

$$F = U - QS,$$ (4)

where $S$ denotes an arbitrary generalized entropy. There is a class of systems for which $F$ determines the stationary behaviors of the systems as well as the stability of the stationary behaviors. In what follows we will consider such systems. More precisely, we will assume that non-stationary behaviors relax to stationary behaviors which involves a decrease of $F$, stationary behaviors of a system correspond to critical points (stationary points) of $F$, and stable stationary behaviors correspond to minima of $F$. In Sec. 2 we will discuss that kind of systems in more detail using nonlinear Fokker-Planck equations. In Sec. 3.2 we will present experimental and theoretical results that have been obtained in studies on neural activity during paced rhythmic finger...
movements and involve a mean field free energy as described by Eq. (4). In Sec. 3.3 we will consider neural activity during self-paced finger movements. We will present some novel experimental findings, assume that there are are free energy measures for task-related populations of neurons, and study the properties of such a free energy in more detail.

2 Stochastic relaxation dynamics

We consider a system described by a scalar random variable defined on the phase space Ω. We assume that the stationary behavior of the system is given by the probability densities $P_{\text{st}}$ that make $F$ (cf. Eq. (4)) stationary. Furthermore, we assume that stable stationary behavior is describe by probability densities $P_{\text{MFE}}$ that correspond to minima of $F$. Then, $P_{\text{st}}$ is defined by $\delta F = 0$ and $\int_{\Omega} P_{\text{st}} \, dx = 1$, where $\Omega$ denotes the phase space of the random variable under consideration and $\delta F$ is the first variation of $F$. $P_{\text{MFE}}$ is a distribution $P_{\text{st}}$ for which $F$ assumes a local or global minimum. The solution of the variational problem $\delta F = 0$ reads $\delta F / \delta P = \mu$, where $\mu$ is a normalization constant and $\delta F / \delta P$ denotes the functional derivative of $F$. We are looking now for an evolution equation for $P(x,t)$ for which $P_{\text{st}}$ and $P_{\text{MFE}}$ are stationary solutions and stable stationary solutions, respectively. We consider classical probability densities $P > 0$ (i.e., we do not consider cut-off solutions) whose evolution is given by

$$\frac{\partial}{\partial t} P(x,t) = \frac{\partial}{\partial x} \left[ M(P) P \frac{\partial}{\partial x} \frac{\partial F}{\partial P} \right],$$

where $M(P) > 0$ describes a mobility coefficient that may depend on $P$. We consider natural, reflective, and periodic boundary conditions [31]. For periodic boundary conditions, we require that $F$ is periodic which implies, for example, that potentials occurring in $F$ are periodic. Then, the probability current $J$ defined by $\partial P / \partial t = -\partial J / \partial x$ vanishes in the stationary case and stationary solutions of Eq. (5) satisfy $\partial \delta F / \partial x \delta P = 0 \Rightarrow \delta F / \delta P = \mu$. That is, the stationary distributions defined by the free energy principle correspond to the stationary solutions of Eq. (5). By means of partial integration, from Eq. (5) it follows that the free energy $F$ evolves like

$$\frac{d}{dt} F[P] = -\int_{\Omega} M(P) P \left[ \frac{\partial}{\partial x} \frac{\delta F}{\delta P} \right]^2 \leq 0.$$

(6)

Furthermore, a stationary solution $P_{\text{st}}$ of Eq. (5) yields $dF/dt = 0$ because $P_{\text{st}}$ satisfies $\delta F / \delta x \delta P = 0$. In turn, the vanishing of the differential quotient $dF/dt$ implies that $\partial \delta F / \partial x \delta P = 0$ for all $x \in \Omega$ because of $M(P)P > 0$. Consequently, we deal with a stationary behavior of the system. In sum, we arrive at the implication

$$\frac{d}{dt} F[P] = 0 \iff \frac{\partial}{\partial t} P(x,t).$$

(7)

Note that Eqs. (5) and (6) can be generalized to the multivariate case, whereas Eq. (7) does not necessarily hold in the multivariate case [32]. Due to Eqs. (6) and (7) stationary solution $P_{\text{st}}$ of Eq. (5) that correspond to maxima and saddle-points of $F$ are unstable. The reason for this is that if $P_{\text{st}}$ corresponds to a maximum or saddle-point of $F$, then in the vicinity of $P_{\text{st}}$ there are probability densities $P' = P_{\text{st}} + \delta P$, where $\delta P$ is small, for which $F[P'] < F[P_{\text{st}}]$. In most cases the $P'$s will correspond to non-stationary solutions of Eq. (5). Due to Eqs. (6) and (7) any non-stationary distribution $P'$ evolves in such a way that $dF < 0$. Consequently, Eq. (5) does not admit for a relaxation process that takes $P'$ back to $P_{\text{st}}$ with $F[P'] < F[P_{\text{st}}]$. Roughly speaking, the stochastic process cannot ”go uphill in the landscape defined by $F$”. As a consequence, only $P_{\text{st}}$ that correspond to local or global minima of $F$ are stable stationary distributions. In order to show that any transient solution converges to a stationary one, we need to prove that $F$ is bounded from below. Then, $F$ is a Lyapunov functional. For finite phase spaces $\Omega$ (reflective or periodic boundary conditions) and energy functionals $U$ that are bounded from below, we can show that $F$ is bounded from below [16, 33]. For infinite phase spaces $\Omega$ (natural boundary conditions) it has been shown for several special cases that $F$ is bounded from below [15, 25, 34, 35]. Eq. (5) is in general nonlinear with respect to $P$. For $F$ given by Eq. (1), a linear energy functional $U$ (i.e., $U[P] = \int_{\Omega} V_0(x) P(x) \, dx$), and a constant mobility coefficient (i.e., $M(P) = M_0$) Eq. (5) becomes
linear and coincides with the well-known Fokker-Planck equation for a single random variable. For this reason and in recognition of the fact that Eq. (5) indeed describes relaxation processes we refer to the evolution equation (5) as a generalized or non-linear Fokker-Planck equation.

3 Paced and self-paced rhythmic movements

3.1 Movement coordination and rhythmic movements

From the perspective of synergetics and complex systems theory, the study of movement coordination is a promising approach to obtain insights into the motor control systems of humans and animals. Movement coordination requires the cooperative behavior of many subsystems. In many cases it has been assumed that this cooperative behavior is a result of self-organization. In particular, it has been argued that movement coordination cannot ignore environmental conditions (walking on ice is different from walking on a carpet). Therefore, feedback from sensory organs is most probably involved in a variety of coordination tasks [36, 37]. From this point of view, rhythmic movements have been studied, where the degree of interlimb coordination can be measured in terms of the degree of synchronization of limb movements [38, 39, 40]. The notion of self-organizing systems has also been used to study movement related cooperative phenomena on the brain level [41, 42]. Phase synchronization of brain activity as known from intracranial recordings in animal studies (see introduction) has been found during simple coordination tasks involving paced and self-paced rhythmic finger movements [38, 43, 44]. In what follows, we briefly review recent work on paced rhythmic finger movements. After that, we will present some novel findings (both experimental and theoretical) regarding self-paced finger movements.

3.2 Paced rhythmic movements

In a series of experiments [43, 44, 45] subjects were asked to tap with their right index fingers along with the frequency of a metronome. They were requested to tap either on the beat (on-beat condition) or between two consecutive beats (off-beat condition). During these experiments the frequency was increased from 1 Hz to about 3 Hz. At low pacing frequencies ($\approx 1$ Hz) both off-beat and on-beat tapping could be performed, whereas at high pacing frequencies ($\approx 3$ Hz) only on-beat tapping could be performed. Consequently, when a subject was asked to tap off-beat with the frequency of a metronome, then there was an involuntary switch from the required off-beat tapping to an on-beat tapping when the metronome frequency exceeded a critical (subject-dependent) value. This nonequilibrium phase transition has been frequently studied in literature [46, 47]. In our experiments, during the performance of on-beat and off-beat tapping, brain activity was measured via magnetoencephalography (MEG) over the whole cortex.

A detailed data analysis showed the following results. The power spectra of single-site recordings are dominated by a peak at the movement frequency (which coincided with the pacing frequency). Irrespective of the task conditions (on-beat vs. off-beat) the phase differences between the oscillatory neural signals and the finger movements are bimodally distributed, see Fig. 1.

The two peaks correspond to two cortical regions that oscillate 180 degrees out of phase, see Fig. 1. The observed bipolarity of cortical activity is in line with other experimental studies on finger movements [48, 49, 50]. The emergence of two sharp peaks in the distributions of the relative phases suggests that within each polarity region brain activity is synchronized with almost vanishing phase differences. This hypothesis can be confirmed by computing so-called phase synchronization variability (PSV) maps [51]. Roughly speaking, in order to draw a PSV map one computes for each MEG signal the probability to find its phase close to one of the two peaks of the bimodal phase distributions shown in Fig. 1. If the probability is high, then phase synchronization variability is low. White and
FIG. 1. Left panels: phase distributions of overall brain activity with peaks close to $\pm \pi/2$; relative phase $[-\pi, \pi]$ measures phase difference between tapping signal and Fourier phase of MEG signals at tapping frequency. Right panels: PSV maps; "white" ("black") regions indicate sites of MEG signals with high degree of synchronization, i.e., with relative phases that can be found with a large probability close to the $+\pi/2$-peak ($-\pi/2$-peak).

black regions in the PSV maps shown in Fig. 1 indicate sites with low phase synchronization variability, that is, high degree of phase synchronization. Whereas regions colored grey correspond to recording sites of MEG signals with high phase synchronization variability. That is, in these regions there is no relevant phase synchronization. In addition, cross-correlation analysis confirms the hypothesis that there are two regions of phase synchronized neural activity, see Fig. 2. In sum, synchronized brain activity can be observed both for off-beat and on-beat tapping. The synchronized state related to off-beat tapping, however, becomes unstable when a control parameter (the pacing frequency) exceeds a critical value.

FIG. 2. Illustrations of cross-correlation coefficients computed from three populations of paired MEG sensors for on-beat condition (upper panel) and off-beat condition (lower panel). We show here the probability to find a cross-correlation coefficient (defined on $[-1, 1]$) rather than the cross-correlation coefficients themselves. Populations $A$ and $B$ are related to inter-cross-correlations of MEG signals from the "white" and "black" polarity regions, respectively, depicted in Fig. 1. Population $C$ reflects intra-cross-correlations between MEG signals taken from "white" and "black" regions. There is a tendency of inter-correlations to be positive and intra-correlations to be negative which hints that the MEG signals of the same polarity region are synchronized with almost zero phase differences.

In view of the experimental findings, we need to model a transition from a bistable to a monostable regime. In the bistable regime, the task-related neuron population exhibits two possible single-peaked stationary phase distributions (related to on- and off-beat tapping). In the monostable regime, there is a unique stable stationary phase distribution (related to on-beat tapping). A model for coupled neural phase oscillators has been proposed which exhibits these features [44]. According to this model, the pacing signal acts on each oscillator as an exter-
nal force. For low frequencies this force corresponds to a bistable potential, whereas for high pacing frequencies it corresponds to a monostable potential, see Fig. 3. More explicitly, the so-called Haken-Kelso-Bunz (HKB) potential \[41, 52\] has been used which involves two control parameters \(a\) and \(b\) and can show such a transition from a double-well potential to a potential with a single minimum. In addition, the neural phase oscillators are assumed to be attractively coupled such that they tend to synchronize their behavior. To this end, a coupling function proposed by Kuramoto has been used \[53\], see Fig. 3. On account of this coupling function, in the low-frequency regime, the phase oscillators cannot occupy both potential minima of the HKB potential at the same time. They either occupy the potential well related to the off-beat tapping or the potential well related to the on-beat tapping. The coupling among the oscillators prevents them from jumping between the wells although there is noise in the system. When control parameters are changed, the HKB potential can change from a bistable to a monostable potential. In particular, in this case the potential minimum related to off-beat tapping vanishes and, consequently, if the oscillator population initially occupies this minimum (off-beat condition) the state of the population becomes unstable, the phase oscillators abandon the unstable state, and the oscillator population eventually settles down into the potential minimum related to the on-beat movement. In short, due to the Kuramoto coupling the proposed oscillator model can exhibit multiple stationary distributions, whereas due to the HKB potential the model can explain the emergence of transitions from off-beat to on-beat tapping.

### 3.3 Self-paced rhythmic movements

In addition to paced rhythmic movements, in our experiments subjects performed self-paced movements \[45, 51\]. In order to fix the tapping frequency, subjects first tapped on-beat with a metronome signal. Then, the metronome was switched off and the subjects continued to tap with the frequency of the metronome. In this self-paced condition, there is no use to distinguish between on-beat and off-beat tapping. Nevertheless, we can investigate whether or not there is a synchronization of brain activity. In line with the literature \[48, 54\], we obtained results similar to those observed during paced tapping. Fig. 4 displays the phase distributions computed from the MEG signals of two subjects performing self-paced finger movements. The phase distributions are bimodal. Again, we resolved the two-peaks of the distributions by means of PSV maps. We found for both subjects two polarity regions of synchronized neural activity, see Fig. 4. They look similar to the polarity regions observed during paced on-beat and off-beat movements, see Fig. 1.

Our objective now is to study the emergence of synchronized brain activity using the theory of...
nonlinear Fokker-Planck equations as outlined in Sec. 2. To this end, we consider a free neuronal phase oscillator \( s(t) = A \cos(\psi(t)) \) described by a time-independent amplitude \( A \) and the phase \( \psi(t) \in [0,2\pi] \) that oscillates with the frequency \( \omega \). In the absence of noise, the evolution equations simply reads \( \dot{\psi} = \omega \). That is, we put \( V_0 = \omega \psi \) in Eq. (2). Then, we model the couplings with other oscillators in terms of mean field interactions involving a sine-function (Kuramoto coupling). Accordingly, \( V_{ik} \) reads \( V_{ik}(\cdot) = -K \cos(\psi - \psi') \) with \( K > 0 \). Finally, we assume that the Boltzmann statistics applies to the population of phase oscillators. Consequently, we use a free energy given by the Eqs. (1) and (2) with reads explicitely

\[
F[u] = -\frac{K}{2} \int_0^{2\pi} \int_0^{2\pi} \cos(\phi - \phi') u(\phi) u(\phi') \, d\phi \, d\phi' - QS_{BGS}[u]
\]

and the nonlinear Fokker-Planck equation (5) becomes

\[
-\frac{\partial}{\partial \psi} \left( \omega - K \int_0^{2\pi} \sin(\psi - \psi') P(\psi', t) \, d\psi' \right) P(\psi, t) + Q \frac{\partial^2}{\partial \psi^2} P(\psi, t) .
\]

Eq. (9) can be simplified by studying the evolution of \( \psi \) in a rotating frame. Introducing the variable \( \phi = \psi - \omega t \), we can transform the nonlinear Fokker-Planck equation (9) into

\[
\frac{\partial}{\partial t} P(\phi, t) = Q \frac{\partial^2}{\partial \phi^2} P(\phi, t) - K \frac{\partial}{\partial \phi} \left( \int_0^{2\pi} \sin(\phi - \phi') P(\phi', t) \, d\phi' \right) P(\phi, t) .
\]

For the sake of convenience, we introduce the cluster amplitude \( r(t) \in [0,1] \) and the cluster phase \( \theta(t) \in [0,2\pi] \) defined by [53]

\[
r(t)e^{i\theta(t)} = \int_0^{2\pi} e^{i\phi} P(\phi, t) \, d\phi .
\]

Then, Eq. (10) reads

\[
\frac{\partial}{\partial t} P(\phi, t) = K r(t) \frac{\partial}{\partial \phi} \sin(\phi - \theta(t)) P(\phi, t) + Q \frac{\partial^2}{\partial \phi^2} P(\phi, t) .
\]

In what follows, we will discuss the stationary solutions \( P_{st}(\phi) \) of Eq. (12) using linear stability analysis [53] and the free energy approach presented in Sec. 2. First of all, from Eqs. (9) and (10) it is clear that for any parameter values \( \omega, K, \) and \( Q \) the uniform distribution is a stationary solution. The uniform distribution corresponds to a stationary cluster amplitude \( r_{st} = 0 \) and describes a desynchronized oscillator population. From Eq. (12) it follows that in general the stationary solution is implicitly defined by

\[
P_{st}(\phi) = N \exp \left\{ \frac{r_{st} K}{Q} \cos(\phi - \theta_{st}) \right\} ,
\]

where \( N \) is given by

\[
N = \left[ \int_0^{2\pi} \exp \left\{ r_{st} K/Q \cos(\phi) \right\} \, d\phi \right]^{-1}
\]

and \( r_{st} \) and
\(\theta_{st}\) are given by Eq. (11). For \(r_{st} > 0\) the distribution \(P_{st}(\phi)\) is single-peaked and nonuniform. Such a nonuniform stationary solution corresponds to a travelling wave solution of Eq. (9). Nonuniform stationary distributions \(P_{st}(\phi)\) indicate that the phase oscillator population is partially synchronized. For the stationary cluster amplitude \(r_{st}\) we obtain from Eqs. (11) and (13) the transcendent equation

\[
\begin{align*}
r_{st} &= N \int_{0}^{2\pi} \cos(\phi-\theta_{st}) \exp \left\{ \frac{r_{st} K}{Q} \cos(\phi-\theta_{st}) \right\} d\phi \\
&= N \int_{0}^{2\pi} \cos(\phi) \exp \left\{ \frac{r_{st} K}{Q} \cos(\phi) \right\} d\phi
\end{align*}
\]

which does not depend on the cluster phase \(\theta_{st}\). Therefore, for \(r_{st} > 0\) the stationary cluster phase can assume any value \(\theta_{st} \in [0, 2\pi]\). For \(K/Q \to \infty\) the stationary solution \(P_{st}(\phi)\) can assume both the uniform distribution \(P_{st}(\phi) = 1/(2\pi)\) with \(r_{st} = 0\) and the delta-distribution \(P_{st}(\phi) = \delta(\phi - \theta_{st})\) with \(r_{st} = 1\) and \(\theta_{st} \in [0, 2\pi]\). The delta-distribution describes complete synchronization of the oscillator population. Linear stability analysis shows that for \(K/Q > 2\) the uniform distribution becomes unstable [53]. Since the free energy functional (8) is bounded from below for any probability density \(u(x)\) [16, 33], it follows from the considerations made in Sec. 2, that for \(K/Q > 2\) there exists a stable stationary nonuniform solution of the form (13) with \(r_{st} > 0\).

The question arises what is the physical interpretation of the observation that the uniform distribution becomes unstable. The answer to this question is that (i) the relaxation dynamics (10) converges to a minimum of the free energy (8) (see Sec. 2) and (ii) for \(K/Q > 2\) the uniform distribution corresponds to a saddle-point of the free energy (8). Let us illustrate the latter point. To this end, we consider the one-parametric family of probability densities

\[
P_{c}(\phi) = N_{c} \exp \left\{ \frac{c K}{Q} \cos(\phi - \theta_{st}) \right\}
\]

with \(c \in [0, 1]\) and \(N_{c} = \left[ \int_{0}^{2\pi} \exp \left\{ c K [Q]^{-1} \cos(\phi) \right\} d\phi \right]^{-1}\). Substituting Eq. (15) into Eq. (8), we obtain

\[
F(c) = -\frac{K}{2} \left\langle \left[ \cos(\phi) \right]_{c}^{2} + K c \cos(\phi) \right\rangle_{c} - Q \ln \left[ \int_{0}^{2\pi} \exp \left\{ \frac{c K}{Q} \cos(\phi) \right\} d\phi \right]
\]

with \(\left\langle g(\phi) \right\rangle_{c} := \int_{0}^{2\pi} g(\phi) P(\phi; c) d\phi\) for any function \(g(\phi)\). Differentiating Eq. (16) with respect to \(c\), gives us

\[
\frac{d}{dc} F(c) = \frac{K^{2}}{Q} \sigma_{\cos}^{2} \left[ c - \left\langle \cos(\phi) \right\rangle_{c} \right]
\]

with \(\sigma_{\cos}^{2} := \left\langle \left[ \cos(\phi) - \left\langle \cos(\phi) \right\rangle_{c} \right]^{2} \right\rangle_{c} \geq 0\). In particular, the free energy is stationary at the uniform distribution \((c = 0)\), that is, for \(c = 0\) we have \(dF/dc = 0\) because of \(c - \langle \cos(\phi) \rangle_{c} = 0\) for \(c = 0\). In order the check whether this critical point corresponds to a maximum or a minimum, we differentiate \(F\) with respect to \(c\), again. Thus, we obtain

\[
\frac{d^{2}}{dc^{2}} F(c) = \frac{K^{2}}{Q} \left\{ \sigma_{\cos}^{2} \left( 1 - \frac{K}{Q} \sigma_{\cos}^{2} \right) + \left( c - \left\langle \cos(\phi) \right\rangle_{c} \right) \frac{d}{dc} \sigma_{\cos}^{2} \right\}.
\]

For the uniform distribution \((c = 0)\) we get \(\sigma_{\cos}^{2} = 1/2\). Consequently, Eq. (18) reads

\[
\frac{d^{2}}{dc^{2}} F(c) \bigg|_{c=0} = \frac{K^{2}}{2Q} \left( 1 - \frac{K}{2Q} \right). \tag{19}
\]

Then, we can read off from Eq. (19) that for \(K/Q > 2\) the free energy \(F(c)\) has a maximum at the uniform distribution with respect to the family \(P_{c}(\phi)\) which implies that \(F[P]\) (for arbitrary \(P\)) has a maximum or a saddle-point at \(P = 1/(2\pi)\) for \(K/Q > 2\).

Finally, we would like to mention that this model nicely illustrates the notion that collective behavior emerges when \(U\) dominates the entropy term, whereas collective behavior vanishes when the entropy term dominates \(U\). Recall that for the uniform distribution the BGS entropy assumes its maximum value \(S_{\text{max}}\). In addition, \(U\) vanishes. Therefore, in the absence of collective behavior, that is, for \(K/Q < 2\), the expression \(QS\) clearly dominates...
Then, the ratio $K/Q$ changes such that $K$ increases or $Q$ decreases. In any case, the expression $QS_{\text{max}}$ does not increase due to this change of the parameters. For $K/Q > 2$ we deal with a stable stationary nonuniform distribution. Consequently, we have $S < S_{\text{max}}$. In addition, $U$ becomes finite. The reason for this is that $U$ can be expressed in terms of $r(t)$ defined in Eq. (11) like

$$U[P] = -\frac{K}{2} \int \int \cos(x-y)P(x)P(y) \, dx \, dy = -\frac{K r^2}{2}.$$  \hspace{1cm} (20)

Consequently, for $K > 0$ and $r_{\text{st}} > 0$ we have $U \neq 0$. Then, as far as the pair $(U, S)$ of thermodynamic state variables is concerned, the emergence of the collective behavior implies a transition from $(U = 0, S = S_{\text{max}})$ to $(U \neq 0, S < S_{\text{max}})$. In this sense, collective behavior emerges when the interaction term becomes more relevant with respect to the entropy term.

4 Conclusion

Stochastic properties of many body systems with identical subsystems have been described in terms of probability densities of single subsystems. We have proposed a generalized Fokker-Planck equation that determines the evolution of these single subsystem probability densities. We have argued that nonlinearities occurring in this evolution equations reflect interactions between subsystems. By means of free energy measures we have shown that transient solutions converge to stationary ones. Therefore, the proposed generalized Fokker-Planck equation describes relaxation processes.

We have provided evidence for the hypothesis that paced tapping involves a multistable stochastic motor control system. Since a conventional linear Fokker-Planck equation usually only admits for a unique stationary solution, we described the multistable motor control system in terms of a nonlinear Fokker-Planck equation based on a subsystem-subsystem interaction term as proposed by Kuramoto and a bistable potential as proposed by Haken, Kelso, and Bunz. In a particular parameter range, this nonlinear Fokker-Planck equation can exhibit multiple stationary probability distributions. The need to introduce subsystem-subsystem interactions can be viewed as a hint that multistable motor control systems consist of many subsystems that are distributed in space but interact with each other. We have also reported from synchronized brain activity observed during self-paced tapping. We have illustrated that in the presence of noise synchronization can arise from mutual couplings between neurons. The stochastic properties of self-paced and paced rhythmic finger movements suggest that task-related neural populations are spatially extended entities and that the population members act in concert in order to produce coordinated movements.

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